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STUDY OF INVERSE PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Declaration

I certify that this thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis (or any part of it) has not been submitted for a higher degree to any other university or institution.

Aseel Titi Signature ....

Abstract

The inverse problem of determining a coefficient (possibly discontinuous) of principal part for parabolic equations has important applications in a large fields of applied science such as heat conduction and hydrology. Hence, the major objective of this thesis is to review the recent results about inverse parabolic problems with discontinuous principal coefficient concerning uniqueness, stability and existence of solution to these problems.

We start by presenting some popular inverse problems in partial differential equations and indicate their applications. Afterward, in part two we study in particular the inverse problems of parabolic partial differential equations. In part three, we review some numerical methods for solving these problems. In part four, we move towards studying these problems with discontinuous principal coefficient, where in this part the uniqueness of recovery of the discontinuous conductivity of a parabolic equation is presented and a numerical solution for an inverse diffusion problem is reviewed.

Finally, the solution of some inverse parabolic problems using Adomian decomposition method is studied and we will try to solve these problems with discontinuous principal coefficient using the mentioned method.

Keywords: inverse problems, inverse parabolic problems, discontinuous principal coefficient, ADM.

الملخص

الهدف الرئيسي من هذه الرسالة هو مراجعة النتائج المتعلقة بالمشاكل العكسية المكافئة ذات المعامل غير المتصل لما لهذه المسائل من تطبيقات هامة في العلوم التطبيقية مثل : التوصيل الحراري والهيدرولوجيا.

في بداية هذه الدراسة نقوم بتقديم بعض المسائل العكسية المشهورة في المعادلات التفاضلية الجزئية ونشير إلى تطبيقاتها. بعد ذلك، قمنا بدراسة المسائل العكسية للمعادلات التفاضلية المكافئة بشكل خاص.

في الفصل الثالث يتم مراجعة بعض أنظمة الحل العددية لهذه المسائل. ومن ثم في الفصل الرابع نقوم بدراسة المسائل العكسية للمعادلات التفاضلية المكافئة ذات المعامل المنفصل.

في النهاية نقوم بدراسة حل هذه المسائل العكسية المكافئة بإستخدام طريقة أدومين للفصل.

كلمات البحث: المسائل العكسية، المسائل العكسية المكافئة، المعامل المنفصل، طريقة أدومين للفصل.

Contents

Preliminaries

In this chapter we present some basic concepts in PDEs and functional analysis that we need in this thesis. We begin with the following definitions.

0.1 Definitions

Definition 0.1.1. [14] *Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set*

$$L^p(\Omega) = \{f:\ \Omega \longrightarrow \mathbb{R};\ f \text{ is measurable and } |f|^p \in L^1(\Omega)\}$$

with

$$||f||_{L^p} = ||f||_p = [\int_\Omega |f(x)|^p \, d\mu]^{\frac{1}{p}},$$

where $L^1(\Omega)$ denotes the space of integrable functions from Ω into $\mathbb{R}$.

Definition 0.1.2. [14] *We set*

$$L^\infty(\Omega) = \{f:\ \Omega \longrightarrow \mathbb{R};\ f \text{ is measurable and there is a constant } C \text{ such that } |f(x)| \le C\}$$

with

$$||f||_{L^\infty} = ||f||_\infty = \inf\{C;\ \ |f(x)| \le C \text{ almost everywhere on } \Omega\}.$$

Now, we introduce Sobolev space $H_k^p(\Omega)$ for open sets Ω in $\mathbb{R}^n$ where k is a non-negative integer.

Definition 0.1.3. *(Sobolev Spaces) The Sobolev space $H_k^p(\Omega)$ is the space of all locally summable functions $u : \Omega \longrightarrow \mathbb{R}$ such that for every multi index α with $|\alpha| \le k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$.*
On $H_k^p(\Omega)$ we shall use the norm

$$||u||_{k,p} = (\sum_{|\alpha| \le k} \int_\Omega |D^\alpha u|^p \ \ dx)^{\frac{1}{p}},\ if\ 1 \le p < \infty,$$

$$||u||_{k,\infty} = \sum_{|\alpha|\leq k} ess\sup |D^\alpha u|\,,\ if\ p=\infty.$$

Definition 0.1.4. *(special case p=2) In the special case where $p = 2$, we define the Hilbert-Sobolev space $H_k(\Omega) = H_k^2(\Omega)$. The space $H_k(\Omega)$ is endowed with the inner product*

$$< u, v >_{H_k} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u D^\alpha v\ dx.$$

In the next definition we adopt the following notations: D is a bounded $(n+1)$-dimensional domain in $\mathbb{R}^{n+1}$, $(x,t) = (x_1, ..., x_n, t)$ is a variable point in $\mathbb{R}^{n+1}$. D is bounded by a domain B on $t = 0$, a domain B_T on $t = T$, and a mainfold lying in the strip $0 < t \leq T$. We set $B_r = D\bigcap\{t = \tau\}$, $S_r = S\bigcap\{t \leq \tau\}$.

Definition 0.1.5. *(Green's Function)*[10] *A function $G(x,t;\xi,\tau)$ defined and continuous for $(x,t;\xi,\tau) \in \overline{D} \times (D\bigcup B)$; $t > \tau$, is called Green's function of $Lu = 0$ in D if for any $0 \leq \tau < T$ and for any continuous function f on B_r having a compact support, the function*

$$u(x,t) = \int_{B_r} G(x,t;\xi,\tau) f(\xi) d\xi$$

is a solution of $Lu = 0$ in $D\bigcap\{\tau < t \leq T\}$ and it satisfies the initial and boundary conditions

$$\lim_{t\longrightarrow r} u(x,t) = f(x) \quad for \quad x \in \overline{B}_r,$$

$$u(x,t) = 0 \quad on \quad S\bigcap\{\tau < t \leq T\}$$

Definition 0.1.6. *(Toeplitz Matrix) Toeplitz matrix or diagonal-constant matrix is a matrix in which each descending diagonal from left to right is constant.*

Definition 0.1.7. *(Holmgren Class) A function $g(t)$ defined on (α,β) is said to be of a Holmgren class if $g \in C^\infty(\alpha,\beta)$ and for all nonnegative integers n, there exists positive constants C and s such that $|g^n(t)| < Cs^n(2n)!$.*

Definition 0.1.8. *(Generalized solution of the parabolic problems)*[1] *A generalized solution $u \in C([0,T]; H_1(\Omega))$ to the initial boundary value problem (2.0.1)-(2.0.3)with $f = f_0 + div\dot{f}$, f_0, $\dot{f} \in L^2(Q)$ as a function satisfying the integral identity*

$$\int_Q (-u\frac{\partial(a_0 v)}{\partial t} + a\nabla u.\nabla v + (b.\nabla u + cu)v)dQ = \int_Q (f_0 v - \dot{f}.\nabla v)dQ + \int_\Omega a_0 u_0 v(,0),$$

for all test functions $v \in H_1^2(Q)$ that are zero on $\Omega \times \{T\}$ and on $\partial\Omega \times (0,T)$.

Using the above definition of a generalized solution to equation (2.0.1) with lateral Neumann data $a\nabla u.v = \Lambda u$ *on* $\partial_x Q$ we obtain the useful identity

$$\int_Q (-u\frac{\partial(a_0 v)}{\partial t} + a\nabla u.\nabla v + (b.\nabla u + cu)v) = \int_Q (f_0 v - \dot{f}.\nabla u) + \int_{\partial\Omega\times(0,T)} \Lambda uv - \int_\Omega a_0 u_0 v(,0),$$

for all test functions $v \in H_1^2(Q)$ that are equal zero on $\Omega \times \{T\}$.

0.2 Theory

Theorem 0.2.1. *(Trace Theorems)*[1] *For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and any $(n-1)$-dimensional Lipschitz surface $S \subset \overline{\Omega}$ there is a constant $C(S,k,q,p)$ such that for all functions $u \in H_k^p(\Omega)$ we have*

$$||u||_q (S) \leq C\, ||u||_{k,p} (\Omega) \quad when\ 1 \leq kp \leq n,$$

and

$$||u||_{\frac{1}{2}} (S) + ||\bigtriangledown u||_{\frac{-1}{2}} (S) \leq C\, ||u||_1 (\Omega).$$

Theorem 0.2.2. *(Interpolation Theorems)*[1] *There is a constant $C(\Omega)$ such that*

$$|\partial^\alpha u|_m u(\Omega) \leq C\, ||u||_{k+\lambda}^{\frac{(|\alpha|+\mu)}{(k+\lambda)}} (\Omega)\, |u|_0^{1-\frac{(|\alpha|+\mu)}{(k+\lambda)}} (\Omega),$$

$$||u||_{(s)} (\Omega) \leq C\, ||u||_{s_1}^{1-\theta} (\Omega)\, ||u||_{s_2}^{\theta} (\Omega),$$

and

$$||u||_2 (S) \leq C\, ||u||_{\frac{1}{2}} (\Omega),$$

provided that $s = (1-\theta)s_1 + \theta s_2 \neq \frac{-1}{2} - k$, for any $k = 0, 1, 2, ...,\ 0 < \theta < 1$.

Theorem 0.2.3. *(Runge Property) Let ω be a C^{∞} domain and Ω analytic currlinear such that every connected component of $\Omega \backslash \omega$ has a boundary came in common with $\partial\Omega$. Let $0 < \gamma_0 \leq \gamma(x)$ be piecewise analytic on $\bar{\Omega}$. Assume that $u \in H^1(\omega)$ satisfy*

$$L_{\gamma_0} u = 0 \ in \ \omega,$$

then given any compact subset $K \subset \omega$, there exists $U \in H^1(\Omega)$ such that

$$L_{\gamma} U = 0 \ in \ \Omega,$$

and

$$\int_K |\nabla(U-u)|^2 \ dx < \epsilon.$$

The following definition will be needed in the next theorem.

Definition 0.2.1. [11] *A functional $p : X \longrightarrow [0, \infty)$ on a linear space X is said to be positively homogeneous provided*

$$p(\lambda x) = \lambda p(x) \quad for \ all \ x \in X, \ \lambda > 0,$$

and said to be subadditive provided

$$p(x+y) \leq p(x) + p(y) \quad for \ all \ x, \ y \in X.$$

Theorem 0.2.4. *(Hahn Banach Theorem)*[11] *Let p be a positively homogeneous, subadditive functional on a linear space X and Y a subspace of X on which there is defined a linear functional ψ for which $\psi \leq p$ on Y. Then ψ may be extended to a linear functional $\tilde{\psi}$ on all of X for which $\tilde{\psi} \leq p$ on all of X.*

Theorem 0.2.5. *(Fubini's Theorem) If $f(x, y)$ is continuous over the region R defined by $a \leq x \leq b$ and $c \leq y \leq d$, then*

$$\int\int_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx = \int_c^d \int_a^b f(x,y) dx dy.$$

lemma 0.2.1. *(Holder's Inequality) Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then*

$$fg \in L^1,$$

and

$$\int |fg| \leq ||f||_p \, ||g||_{p'} \, .$$

Theorem 0.2.6. *(Lebesgue dominated-convergence Theorem)*[11] *Let* $\{f_n\}$ *be a sequence of functions in* L^1*. Suppose that there is a function* $g \in L^1$ *such that for all* n*,* $|f_n(x)| \leq g(x)$ *almost everywhere on* Ω*. If* $\{f_n\} \longrightarrow f$ *almost everywhere on* Ω*, then* $f \in L^1$ *and* $\lim_{n \longrightarrow \infty} \int_\Omega f_n = \int_\Omega f$*.*

Theorem 0.2.7. *(Property of unique continuation) Assume that* $f \in C^\omega(\Omega)$ *where* Ω *is connected open set in* $\mathbb{R}^n$ *then* f *is determined uniquely in* Ω *if we know* $D^\alpha f(z)$ *for all* $\alpha \in \mathbb{Z}^n$*, or* f *is determined uniquely in* Ω *by it's values in any nonempty open subset of* Ω*, or if* $f = 0$ *in any open subset of* Ω *then* $f = 0$ *in* Ω*.*

Theorem 0.2.8. *(Whitney Extension Theorem) Suppose* r *is a non-negative integer and* $A \subseteq \mathbb{R}^n$ *is closed. Assume for each* $x \in A$ *that there exists a polynomial*

$$P_x(y) = \sum_{|\alpha| \leq r} a_\alpha y^\alpha, \quad \textit{for all } y \in \mathbb{R}^n,$$

(where each $a_\alpha \in \mathbb{R}$*)so that*

$$\lim_{|x-y| \to 0} \left| D^\beta P_y(y) - D^\beta P_x(y) \right| |x-y|^{-(r-|\alpha|)} = 0,$$

uniformly on compact subsets of A *for each multi-index* β *with* $|\beta| \leq r$*.*
Then there is a function $f : \mathbb{R}^n \to \mathbb{R}$ *of class* C^r *so that*

$$D^\beta f(x) = D^\beta P_x(x),$$

for each multi-index β *with* $|\beta| \leq r$ *and each* $x \in A$*.*

0.3 Chebyshev Polynomials of the first kind

The Chebyshev polynomials $T_n(x)$ can be obtained by means of Rodrigue's formula

$$T_n(x) = \frac{(-2)^n n!}{(2n)!} \sqrt{1-x^2} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}.$$

When the first two Chebyshev polynomials $T_0(x)$ and $T_1(x)$ are known, all other polynomials $T_n(x)$, $n \geq 2$ can be obtained by means of the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

The derivative of $T_n(x)$ with respect to x can be obtained from

$$(1-x^2)T_n^{'}(x) = -nxT_n(x) + nT_{n-1}(x).$$

One of the most important properties of the Chebyshev polynomials is their orthogonality on the interval $[-1, 1]$, i.e

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$$

We observe that the Chebyshev polynomials form an orthogonal set on the interval $-1 \leq x \leq 1$ with the weighting function $(1-x^2)^{\frac{-1}{2}}$.
Additional Identities of Chebyshev polynomials is implemented as follow

$$T_{n+1}(x) = \cos[(n+1)\cos^{-1} x],$$

$$\frac{T_{n+1}^{'}}{n+1} - \frac{T_{n-1}^{'}(x)}{n-1} = \frac{2\cos\theta\sin\theta}{\sin\theta} = 2T_n(x), \quad n \geq 2,$$

and

$$\int T_n(x)\ dx = \frac{1}{2}[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}] + C, \quad n \geq 2.$$

0.4 Legendre Polynomials

The Legendre polynomials are defined by Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n, \quad n = 0, 1, 2, ...,$$

for arbitrary real or complex values of the variable x.
One of the most important properties of the Legendre polynomials is their orthogonality on the interval $[-1, 1]$, i.e

$$\int_{-1}^{1} P_m(x)P_n(x) = 0 \quad if\ m \neq n,$$

and

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1} \quad if\ m = n.$$

They can be determined with the aid of the following recurrence formulas:

$$P_0(x) = 1,\ P_1(x) = x,\ P_{i+1}(x) = \frac{2i+1}{i+1}xP_i(x) - \frac{i}{i+1}P_{i-1}(x), \quad i = 1, 2, ...$$

The shifted Legendre polynomials are a set of functions analogous to the Legendre polynomials, but defined on the interval $(0, 1)$. They can be defined as

$$P_n^h(x) = P_n(\frac{2x-h}{h}),$$

and they obey the orthogonality relationship

$$\int_0^1 P_m^h(x)P_n^h(x)dx = \frac{h}{2n+1}\delta_{mn},$$

where δ_{mn} is the Kronecker delta.
The analogue of Rodrigue's formula for the shifted Legendre polynomials is

$$P_n^h(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(\frac{2x-h}{h})^2 - 1]^n.$$

The shifted Legendre polynomials are obtained using the following recurrence formula:

$$P_0^h(x) = 1,\ P_1^h(x) = \frac{2x-h}{h},\ P_{i+1}^h(x) = \frac{(2i+1)(2x-h)}{(i+1)h}P_i^h(x) - \frac{i}{i+1}P_{i-1}^h(x),\ \ i = 1, 2, ...$$

A function $u(x, t)$ of two independent variables defined for $0 \leq x \leq l$ and $0 \leq t \leq \tau$ may be expanded in terms of shifted Legendre polynomials as

$$u(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}P_i^\tau(t)P_j^l(x) = \psi^T(t)\mathbf{A}\phi(x), \qquad (0.4.1)$$

where $$\mathbf{A} = \begin{bmatrix} a_{00}\ldots & a_{0m} \\ . & \\ . & \\ . & \\ a_{n0}\ldots & a_{nm} \end{bmatrix}, \quad \phi(x) = \begin{bmatrix} P_0^l(x) \\ P_1^l(x) \\ . \\ . \\ . \\ P_m^l(x) \end{bmatrix},$$

and

$$\psi(t) = \begin{bmatrix} P_0^{\tau}(t) \\ P_1^{\tau}(t) \\ . \\ . \\ . \\ P_n^{\tau}(t) \end{bmatrix}.$$

Additional identities of shifted legendre polynomials is implemented as follow:

$$P'^{h}_{r+1}(t) - P'^{h}_{r-1}(t) = \frac{2(2r+1)}{h} P_r^h(t), \quad r = 1, 2...,$$

$$\int_0^t P_r^h(t^{'})dt^{'} = \frac{h}{2(2r+1)}[P_{r+1}^h(t) - P_{r-1}^h(t)], \quad r = 1, 2, ...$$

0.5 A priori estimates of the Schauder Type

An a priori estimate is an estimate which can be derived without a rudimentary knowledge that the solutions for which the estimate holds, do in fact exist.
Some a priori estimates can be used to prove the existence of solutions, i.e; if one can prove an a priori estimate for solutions of an equation, then it's often possible to prove that solutions exist using the continuity method or a fixed point theorem.
These estimates are similar to estimates derived by Schauder about the regularity of solutions to linear, uniformly, elliptic partial differential equations.

There are two kinds of Schauder estimates. The first kind is called interior estimates where the holder condition for the solution is given in interior domains a way from the boundary. The second kind is called boundary estimates where the holder condition for the solution is given in the entire domain.[10]
Consider the equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t) \; in \; D + B_T. \tag{0.5.1}$$

The following assumptions as in [10] will be needed:
(A)The coefficients of L are locally Holder continuous (exponent α) in D,

and

$$|a_{ij}|_\alpha \leq K_1, \ \ |db_i|_\alpha \leq K_1, \ \ \left|d^2c\right|_\alpha \leq K_1.$$

(B)For any $(x,t) \in D$ and for any real vector ξ,

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq K_2 \left|\xi\right|^2, \quad (K_2 > 0).$$

(C)f is locally Holder continuous (exponent α) in D and

$$\left|d^2 f\right|_\alpha < \infty.$$

We will now state the interior estimates theorem.

Theorem 0.5.1. [10] *Let (A), (B), (C) hold. There exists a constant K depending only on K_1, K_2 and on n, α such that for any solution u of (0.5.1) in D for which $||u||_0^D < \infty$ and u_1, $D_x u$, $D_x^2 u$, $D_t u$ are locally Holder continuous (exponent α) in D, u must belong to $C^{2+\alpha}$ and*

$$||u||_{2+\alpha} \leq K(||u||_0 + \left|\left|d^2 f\right|\right|_\alpha).$$

0.6 Notations

Let $\Omega \subset \mathbb{R}^n$ be an open set, then the following notations will be used in our thesis.
$C(\Omega)$ is the space of continuous functions on Ω.

$C^k(\Omega)$ is the space of functions k times continuously differentiable on Ω if $k \geq 1$ is an integer and $k - \{k\}$ times holder continuous when k is fractional.

$C^\infty(\Omega) = \bigcap_k C^k(\Omega)$(Stands for the set of functions with continuous partial derivatives of any order).

$C_0(\Omega)$ is the space of continuous functions on Ω with compact support in Ω, i.e, which vanish outside compact set $K \subset \Omega$.

$C_0^\infty(\Omega) = C^\infty(\Omega) \bigcap C_0(\Omega)$(Denotes the class of infinitely smooth functions in Ω with compact support).

$C^{k,\alpha}(\Omega)$: $k \in \mathbb{N}$, $0 < \alpha < 1$, denotes the linear space of functions in $C^k(\Omega)$ whose k-th order partial derivatives are Holder continuous, i.e, for all $\beta \in \mathbb{N}$ with $|\beta| = k$ there exist constants $\Gamma_\beta > 0$ such that for all $x,\ y \in \Omega$ we have

$$\left| D^\beta u(x) - D^\beta u(y) \right| \leq \Gamma_\beta \left| x - y \right|^\alpha ,$$

with norm

$$||u||_{C^{k,\alpha}(\Omega)} = ||u||_{C^k(\Omega)} + \max_{|\beta|=k} \sup_{x,y\in\overline{\Omega}} \frac{\left| D^\beta u(x) - D^\beta u(y) \right|}{\left| x - y \right|^\alpha}$$

If $f \in C^1(\Omega)$, it's gradient is defined by

$$\nabla f = (\frac{\partial f}{\partial x_1},\ \frac{\partial f}{\partial x_2}, ...,\ \frac{\partial f}{\partial x_n}).$$

If $f \in C^k(\Omega)$ and $\alpha = (\alpha_1,\ \alpha_2, ...,\ \alpha_n)$ is a multi-index of length $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, less than k, we write

$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} ... \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f.$$

Chapter 1

Inverse Problems

In this chapter we organize some inverse problems and mention their applications. Given a full characterization of a physical system we can predict the outcome of some measurements and we arrive to a forward problem or a direct problem. Using the actual result of some measurements to infer the values of the parameters that describe the system we get an inverse problem. In other words, inverse problems could be described as problems where the answer is known but not the question or where the results or consequences are known but not the reason.
Inverse problems have many applications in many branches of science and mathematics, including computer graphics and computer vision, natural language processing, machine learning, statistics, statistical inference, geophysics, medical imaging, astronomy, physics and many other fields.

1.1 Inverse problem of gravimetry

The gravity field u around a planet which can be measured by the gravity force ∇u and which is generated by the mass distribution f is a solution to the Poisson equation

$$-\Delta u = f \ \ in \ \ \mathbb{R}^3, \ where \lim_{|x|\to+\infty} u(x) = 0.$$

The direct problem of gravimetry is to find the values of the gravity field u around the planet given the distribution of mass f inside the planet, but there are different distributions of mass that give exactly the same gravity field in the space outside the planet. This is a well posed problem it's solution exists for any distribution that is zero outside a bounded domain Ω, it's unique and stable. So that, the problem can be solved numerically by using finite difference schemes. In fact the solution is given by the Newtonial potential

$$u(x) = \int_{\Omega} k(x-y) f(y) dy, \quad k(x) = \frac{1}{4\pi |x|}.$$

The inverse problem of gravimetry is to infer the mass distribution f from observations of the gravity field u or more physically ∇u on Γ which is a part of the boundary $\partial\Omega$.
An important property of the inverse problem of gravimetry is it's ill-posedness, which creates many mathematical and numerical difficulties. In fact, the ill-posedness comes from the nonuniqueness of the mass distribution.
This problem has many applications in recovering the interior of the earth by translating results of measurements of the gravity field. Another application is in gravitational navigation, where one can measure gravity field (by satellites), and then find the function f that produces this field, and then use these results to navigate aircrafts.

1.2 Inverse scattering

The direct scattering problem is to determine the scattered wave for a given object based on the properties of the scatterer. In other words, we need to calculate the total field for acoustic and electromagnetic scattering problems with corresponded wave numbers. On the other hand, the inverse problem is to find a shape of a scattering object given the intensity (and phase) of sound or electromagnetic waves scattered by this object.
In summary, the task of direct scattering theory is to determine the relation between the input and output waves given the details about the scattering target while the task of inverse scattering theory is to determine characteristics of the target, given sufficiently many input and output pairs.
We consider solutions to the Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0 \quad in \quad \mathbb{R}^3 \backslash \bar{D},$$

subject to the Dirichlet(for a soft obstacle)boundary condition

$$u = 0 \quad on \quad \partial D,$$

or Neumann(for a hard obstacle)boundary condition

$$\frac{\partial u}{\partial \nu} + bu = 0 \ \ on \ \ \partial D,$$

where our space is $\mathbb{R}^3$ and the obstacle $D \subset \mathbb{R}^3$ is a closed bounded set.

We are looking for a family of solutions u such that the solution u is decomposed into incident field u^i and scattered field u^s that is

$$u(x) = u^i(x) + u^s(x),$$

where

$$u^i(x) = \exp(ik\zeta x),$$

and

$$u^s(x) = \frac{\exp ik\,|x|}{|x|} A(\sigma, \zeta; k) + o(\frac{1}{|x|}).$$

For a given obstacle, these equations uniquely determine the function $A(\sigma, \zeta; k)$, called the scattering amplitude or the far field pattern. The inverse scattering problem is to find a scatterer from far field pattern.

Inverse scattering problems arise in many applications including computer tomography, seismic and electromagnetic exploration in geophysics. We know about the interior structure of the earth by solving the inverse problem of determining the sound speed by measuring travel times of seismic waves, the structure of DNA by solving inverse X-ray diffraction problems and the structure of the atom and it's components from studying the scattering when materials are bombarded with particles. Medical imaging uses scattering of X-rays, ultrasound waves and electromagnetic waves to make images of the human body which is of valuable help with medical diagnosis. The oil exploration industry uses the reflection of seismic waves in oil prospecting.

1.3 The inverse conductivity problem

The conductivity equation for electric potential u is

$$div(a\nabla u) = 0 \ \ in \ \ \Omega,$$

subject to the Dirichlet data at the boundary

$$u = g \quad on \quad \partial\Omega,$$

where a is a scalar function that is measureable and bounded.

The direct problem is to find u given a and g. This problem is well posed, i.e the solution exists for Lipschitz g. For the inverse conductivity problem we consider the additional data

$$h = \frac{\partial u}{\partial \nu} \; on \; \Gamma,$$

where Γ is a part of the boundary.
Hence, the inverse problem is to find a and u for one g (one boundary measurements) or for all g (many boundary measurements). The inverse conductivity problem is nonlinear because both a and u are unknown.

This problem applies to a variety of situations when someone need to find interior characteristics of a physical body from boundary measurements and observations. The inverse conductivity problem can be used as a fundamental mathematical model for the electrical impedance tomography which is a good method for predicting the interior of the human body by surface electromagnetic measurements. Also, the same mathematical model works well for magnetometric methods in geophysics, mine and rock detection, and underground water search.

1.4 Tomography (Integral geometry)

The aim of integral geometry is to find a function f given the integrals $\int_\gamma f d\gamma$ over a family of manifolds $\{\gamma\}$.
Some mathematical theorems in integral geometry form the foundation of tomography. The goal of tomography is to recover the interior structure of a body using external measurements. X-ray computed tomography is the most basic type of tomography where the goal of this technique is to get a picture of the internal structure of an object by X-raying the object from many different directions.

As an X-ray passes through a patient, it is attenuated so that it's intensity is reduced. This depends upon what material the ray passes through: it's intensity is reduced more when passing through bone than when passing through muscle. To reconstruct an image of the body from a set of X-ray measurements it's very important to measure exactly how different materials absorb X-rays.
Now, when the X-ray enters the body it does so in a straight line with intensity I_{start} and when it leaves it will have intensity I_{finish}. Hence, the attenuation(reduction in the intensity)is given by

$$I_{finish} = I_{start} \exp^{(-R)},$$

where

$$R = \int u(s)\ ds, \quad u(s) : optical\ density\ of\ the\ material.$$

Then, the inverse problem is to find the density $u(s)$ given the attenuation of the intensity.

1.5 Inverse spectral problems

Consider the eigenvalue problem

$$-\Delta u + \lambda u = 0 \quad in \quad D,$$

subject to the Dirichlet boundary condition

$$u = 0 \quad on \quad \partial D. \tag{1.5.1}$$

The main objective in spectral theory of differential operators is relations between geometric data (coefficients of (1.5.1), shape of the boundary) and spectral data (eigenvalues or eigenfunctions of (1.5.1)).

The direct problem is to find the eigenvalues $\{\lambda_k\}$ and solution of (1.5.1) given the coefficients of the equation (1.5.1), while the inverse problem is to recover the geometric data from suitable spectral data.

Both problems have numerous applications in physics. The physical models include vibration of elastic bodies, acoustics, electromagnetic waves and quantum mechanics.

In PDEs, we divide inverse problems into three main areas including: inverse problems in elliptic, hyberbolic, and parabolic.

In a direct well posed problem we need to find a solution for a given partial differential equation subject to initial and boundary conditions. But in inverse problems, the PDE and initial conditions or boundary conditions are not fully specified but instead some additional information is available.

So, when we deal with inverse mathematical physics problems we can speak of coefficient inverse problems, boundary inverse problems, geometric inverse problems in which the domain is unknown, and evolutionary inverse problems in which initial conditions are unknown.

Inverse problems for partial differential equations are nonlinear, and most of them are illposed in the sence of Hadamard. Illposedness must be handled either by using a priori information which stabilize the problem or by using appropriate numerical methods called regularization techniques.
But these problems represents the most popular mathematical models of recovery of unknown physical, geophysical, or medical objects from exterior observations. Hence, it's very important to study the recent results about uniqueness and stability of these problems.

Chapter 2

Inverse parabolic problems

In this chapter we consider the second order parabolic equation in multi dimensional case:

$$a_0 \frac{\partial u}{\partial t} - div(a\nabla u) + b.\nabla u + cu = f \quad in \quad Q = \Omega \times (0,T), \tag{2.0.1}$$

subject to the initial condition

$$u = u_0 \qquad on \quad \Omega \times \{0\}, \tag{2.0.2}$$

and the lateral boundary condition

$$u = g \qquad on \quad \partial\Omega \times (0,T), \tag{2.0.3}$$

where Ω is a bounded domain in the space $\mathbb{R}^n$ with the C^2-smooth boundary $\partial\Omega$. Here a is a symmetric strictly positive matrix function with entries in $L^\infty(Q)$, b is a real-valued vector function with the same regularity properties; and a_0, c are real-valued $L^\infty(Q)$-functions, $a_0 > \epsilon > 0$.

It has been proved that if Ω is bounded and in $C^{2+\lambda}$, $0 < \lambda < 1$, the functions a_0, b, c, f belongs to $C^{\lambda,\frac{\lambda}{2}}(\overline{Q})$, the coefficient matrix a in $C^{1+\lambda,\frac{\lambda}{2}}(\overline{Q})$, then the initial boundary value problem (2.0.1)-(2.0.3) has a unique solution $u \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{Q})$ provided the boundary data $g \in C^{2+\lambda,1+\frac{\lambda}{2}}(\partial\Omega \times [0,T])$ and the following conditions are satisfied: $g = 0$, $a_0\frac{\partial g}{\partial t} = f$, on $\partial\Omega \times \{0\}$.
Also, when all coefficients are only in $C(\overline{Q})$, $a \in C^1(\overline{Q})$ and $f \in L^\infty(Q)$ then an unique solution u can be obtained in the anisotropic Sobolev spaces $H_p^{2,1}(Q)$, $1 < p < \infty$.

If the coefficients are in $L^\infty(Q)$ one has an unique generalized solution u that belongs to $H_2^{1,0}(Q)\bigcap C^{\mu,\frac{\mu}{2}}(\overline{Q})$ for some $\mu\in(0,1)$.

Solutions to linear parabolic equations have important properties concerning the positivity principle and maximum principle which can be formulated as follows.

Theorem 2.0.1. [1]*(The positivity principle)*
If $f_j=0$, $f_0\geq 0$ on Q, $g\geq 0$ on $\partial\Omega\times(0,T)$, $u_0\geq 0$ on Q, then $u\geq 0$ on Q. In addition, if $u(x,t)>0$, then $u(y,s)>0$ when $y\in\Omega,\ t<s\leq T$.

Theorem 2.0.2. [1]*(Maximum Principle)*
If $f_j=0$ and $c\geq 0$ in Q, then $u\leq\max u$ over ∂Q_T provided $\max u\geq 0$.

We are going to present two results about direct problems where the coefficients are independent of time. Let $A=-div(a\nabla)+b.\nabla+c$.

Theorem 2.0.3. [1]*(Analyticity with respect to t)*
Assume that the coefficients of A do not depend on t; $a_0,\ \nabla a\in C(\bar{\Omega})$; $b,c\in L^\infty(\Omega)$; the right hand side of (2.0.1) $f=0$; and the Dirichlet data $g\in C^2(\partial\Omega\times\mathbb{R}^+)$ are (complex) analytic in a sector containing the ray $(T-\epsilon,\infty)$. Then a solution to the first boundary value problem (2.0.1), (2.0.2),(2.0.3) is complex analytic with respect to t in a smaller sector. There is C depending only on Ω, $H_1^\infty(\Omega)$-norms of a_0, a, b, c, and the ellipticity constant of A such that for a solution u to the parabolic boundary value problem (2.0.1), (2.0.2), (2.0.3) is with $T=\infty$, $f=0$, $u_0=0$, and $g\in H_{\frac{1}{2}}(\partial\Omega\times(0,\tau))$ that doesn't depend on $t\in(\frac{\tau}{2},\infty)$, one has

$$||u(,t)||_{(1)}(\Omega)\leq C\,||g\exp(Ct)||_{(\frac{1}{2})}(\partial\Omega\times(0,\tau)),\ t\in S.$$

Theorem 2.0.4. [1] *Assume that the coefficients of A don't depend on t; $c\geq 0$, and g doesn't depend on $t\in(\frac{\tau}{2},\infty)$, $g\in C^2(\partial\Omega\times\mathbb{R}^+)$,$||g||_{(\frac{1}{2})}(\partial\Omega\times(0,\tau))\leq 1$. Let u^0 be the solution to the Dirichlet problem $Au^0=0$ in Ω, $u^0=g(,\tau)$, then there are C and $\theta>0$ depending only on Ω, τ, on the norms of the coefficients and of ∇a in $L^\infty(\Omega)$, and on the ellipticity constant of A such that $||u(,t)-u^0||_\infty(\Omega)\leq C\exp(-t\theta)$, when $\frac{\tau}{2}<t$.*

Some basic solvability and regularity results about the direct problem (2.0.1)-(2.0.3) are described in the following theorem.

Theorem 2.0.5. [1] *(i) (Weak Solutions) Assume that* $f_j \in L^2(Q)$, $u_0 \in L^2(\Omega)$, g, $\frac{\partial g}{\partial t} \in L^2((0,T); H_{\frac{1}{2}}(\partial\Omega))$. *Then there is a unique (generalized) solution* $u \in C([0,T]; H_1(\Omega))$ *to the initial boundary value problem (2.0.1)-(2.0.3).*
(ii) (Regularity) If $f_j \in L^\infty(Q)$ *and* $u \in C^{\lambda,\frac{\lambda}{2}}(\Gamma)$, *where* Γ *is an (open) part of* $\overline{\partial Q_T}$ *and* $\partial Q_T = Q \bigcap \{t < T\}$, $0 < \lambda < 1$ *then* $u \in C^{\mu,\frac{\mu}{2}}(Q \bigcup \Gamma)$ *with some* $\mu \in (0,1)$. *If* $\partial\Omega \in C^{2+\lambda}$, a_0, a^{jk}, $\partial_m a^{jk}$, b_k, c, $f_0 \in C^{\lambda,\frac{\lambda}{2}}(\overline{Q^0})$, $f_j = 0$, $u_0 \in C^{2+\lambda}(\Omega \bigcap \overline{Q^0})$, $g \in C^{2+\lambda,1+\frac{\lambda}{2}}(\partial\Omega \times [0,T] \bigcap \overline{Q^0})$, *where* Q^0 *is a subdomain of* Q *and the following compatibility conditions are satisfied:* $u_0 = g$, $\frac{\partial g}{\partial t} + Au_0 = f_0$ *on* $\partial\Omega \times \{0\}$, *then* $u \in C^{2+\lambda,1+\frac{\lambda}{2}}(Q^0 \bigcup \Gamma^0)$, *where* Γ^0 *is an open part of* $\partial Q \bigcap \overline{Q^0}$.

In inverse problems one is looking for one or several coefficients of equation (2.0.1) or for the source term f, when in addition to the initial and lateral boundary data (2.0.2), (2.0.3) we are given either the final data

$$u = u_T \quad on \quad \Omega \times T, \tag{2.0.4}$$

or the lateral (Neumann) data

$$a\nabla u.\nu = h \quad on \quad \Gamma \times (0,T), \tag{2.0.5}$$

where Γ is a part of $\partial\Omega$ and ν is the unit outward normal to $\partial\Omega$.

2.1 Final overdetermination

In this section we consider the uniqueness, stability and existence of solution to the inverse problem with additional data at the final moment of time T. In the next theorem assume that $\partial\Omega$ is of class $C^{2+\lambda}$ and λ is any number in $(0,1)$.

Theorem 2.1.1. [1] *Assume that* $f = \alpha F$, *where* $F \in C^\lambda(\bar{\Omega})$, $\frac{\partial F}{\partial t} = 0$, *and that the coefficients of equation (2.0.1) and the weight function* α *are given and that they satisfy the following conditions:*

$$a,\ b\ don't\ depend\ on\ t;\ c \geq 0,\ \frac{\partial c}{\partial t} \leq 0\ on\ Q;\ \frac{\partial a_0}{\partial t},$$

$$\frac{\partial c}{\partial t},\ \alpha,\ \frac{\partial \alpha}{\partial t} \in C^{\lambda,\frac{\lambda}{2}}(\bar{Q}),$$

$$\alpha \geq 0, \frac{\partial \alpha}{\partial t} \geq 0 \ on\ Q;\ \epsilon < \alpha \ on\ \Omega \times \{T\}. \tag{2.1.1}$$

Then, there is a constant C depending only on $|| \ \ ||_{\lambda,\frac{\lambda}{2}}$-norms of the coefficients of the parabolic equation (2.0.1) and of α and on the positive number ϵ such that any solution (u, F) to the inverse source problem (2.0.1), (2.0.2), (2.0.3), (2.0.5) satisfies the inequality

$$||u||_{\lambda+2,\frac{\lambda}{2}+1}(Q)+||F||_{\lambda}(\Omega) \leq C(||u_0||_{\lambda+2}(\Omega)+||u_T||_{\lambda+2}(\Omega)+||g||_{\lambda+2,\frac{\lambda}{2}+1}(\partial\Omega\times(0,T))). \tag{2.1.2}$$

Moreover, a solution (u, F) with the finite norms on the left hand side of (2.1.2) exists for any data g, u_0, u_T with the finite norms on the right hand side of (2.1.2).

Proof. Assume that $u_0 = 0$, $g = 0$ and consider the (integral) equation

$$F - \alpha^{-1}(,T)a_0(,T)\frac{\partial u(,T;F)}{\partial t} = \alpha^{-1}(,T)A(,T)u_T, \tag{2.1.3}$$

where $u(x, t; F)$ denotes the solution to the direct parabolic problem (2.0.1)-(2.0.3) with $f = \alpha F$ and $A = -div(a\nabla) + b.\nabla + c$ is the second-order elliptic operator.
Equation (2.1.3) implies that $A(,T)u_T = A(,T)u(,T;F)$ on Ω, and then $u_T = u(,T;F)$ by uniqueness in the Dirichlet problem.
Due to the smoothing properties of diffusion equation with respect to time, the operator $F \to \frac{\partial u(,T;F)}{\partial t}$ in the space $C^l(\bar{\Omega})$ is continuous. Hence, equation (2.1.3) is fredholm, and for any δ there is $C(\delta)$ such that

$$\left|\left|\frac{\partial u(,T;F)}{\partial t}\right|\right|_l(\Omega) \leq \delta\,||F||_l(\Omega) + C\,||F||_0(\Omega). \tag{2.1.4}$$

Transferring the term with $\frac{\partial u}{\partial t}$ into the right side of (2.1.3) applying (2.1.4) and the triangle inequality, and using the regularity of the coefficients and of α, we get

$$||F||_l(\Omega) \leq \delta\,||F||_l + C\,||F||_0 + C\,||u_T||_{l+2}(\Omega).$$

Choosing $\delta < 1$, we obtain the estimate (2.1.2)with the additional term $||F||_0(\Omega)$ on the right side.

To prove the uniqueness it's sufficient to show that $u = 0 \ in\ Q$, provided that the boundary data (2.0.2), (2.0.3), (2.0.4) are zero.

Assume that F is not zero on Ω. Let $f^+ = \frac{f+|f|}{2}$ and $f^- = \frac{-f+|f|}{2}$. Since $\alpha \geq 0$, we have $(\alpha F)^+ = \alpha F^+$, and $(\alpha F)^- = \alpha F^-$.

Since $F \in C^\lambda(\bar{\Omega})$, the functions F^+, F^- have the same regularity, F^- can't be zero since if $F^- = 0$, then $F \geq 0$ and by the positivity principle $u > 0$ *on* $\Omega \times \{T\}$ which contradicts the condition (2.0.4) to be zero. Simillarly, the case $F^+ = 0$ is not possible.

Let u^+, u^- be solutions to the parabolic problem (2.0.1), (2.0.2), (2.0.3) with zero initial and lateral boundary data and with the source term αF^+, αF^-. By theorem (2.0.5) these solution exist and are in $C^{2+\lambda,1+\frac{\lambda}{2}}(\bar{Q})$. By positivity principle, u^+, $u^- > 0$.

We claim that
1. $\frac{\partial u^+}{\partial t} > 0$ *on* Q *and on* $\Omega \times \{T\}$.
2. $\frac{\partial u^-}{\partial t} > 0$ *on* Q *and on* $\Omega \times \{T\}$.

To obtain, the first claim: by using finite difference with respect to t and an estimate for the solution of the parabolic problems before the solution is known to exist, one can show that $w^+ = \frac{\partial u^+}{\partial t} \in C^{2+\lambda,1+\frac{\lambda}{2}}(\bar{Q})$.
Differentiating equation (2.0.1) with respect to t we get

$$a_0 \frac{\partial w^+}{\partial t} - div(a \nabla w^+) + b.\nabla w^+ + (c + \frac{\partial a_0}{\partial t}) w^+ = \frac{\partial \alpha}{\partial t} F^+ - \frac{\partial c}{\partial t} u^+ \geq 0 \; in \; Q,$$

by conditions (2.1.1).

From the equation for u^+ at $t = 0$ we have $w^+ = \frac{\partial u^+}{\partial t} = -A0 + \alpha a_0 F^+ \geq 0$ *on* Ω. Hence, $w = 0$ *on* $\partial\Omega \times (0, T)$. By positivity principle, $\frac{\partial u^+}{\partial t} = w^+ > 0$. A proof of the second one is similar.

Now u^+ has a maximum point $x^0 \in \bar{\Omega}$ since $u^+ = 0$ *on* $\partial\Omega \times \{T\}$ and $u^+ \in C(\bar{\Omega} \times \{T\})$. Since $u^+ \geq 0$ and $\frac{\partial u^+}{\partial t}$, $\frac{\partial u^-}{\partial t} > 0$ this maximum is positive, and $x^0 \in \Omega$. Also x^0 is a maximum point of u^- *on* $\Omega \times \{T\}$ since $u = 0$ *on* $\partial\Omega \times \{T\}$.

From the definition we have that either (a) $\alpha F^+(x^0, T) = 0$ or (b) $\alpha F^-(x^0, T) =$

0. In case (a) using the above two claims we obtain $Au^{+}(x^0,T) = 0 - a_0\frac{\partial u(x^0,T)}{\partial t}$. By using the maximum principles for elliptic equations we conclude that u^+ is constant near x^0 which gives a contradiction with $Au^+ < 0$ *on* $\Omega \times \{T\}$ near x^0. Similarly, we arrive at a contradiction in case (b). The contradiction shows that the initial assumption was wrong, so $u = 0$.

□

Now we will study identification of a time-independant coefficient c.

Theorem 2.1.2. [1] *Consider the initial boundary value parabolic problem (2.0.1)-(2.0.3) with* $a_0 = a = 1$, $b = 0$, $f = 0$, $u_0 = 0$, *and given* $g \in C^{(2+\lambda, 1+\frac{\lambda}{2})}(\partial\Omega \times [0,T])$ *from the additional final over determination (2.0.4) where the coefficient* $c = c(x) \in C^{\lambda}(\bar{\Omega})$ *is unknown then we have the following results:-*
(i)(Uniqueness) If

$$g \geq 0,\ \frac{\partial g}{\partial t} \geq 0 \text{ and } \frac{\partial g}{\partial t} \neq 0 \text{ on } \partial\Omega \times (0,T), \tag{2.1.5}$$

then a solution (u, c) *to the inverse coefficient problem is unique.*
(ii)(Stability) If (u_1, c_1), (u_2, c_2) *are solutions of the inverse problem with the data* g_1, u_{01}, g_2, u_{02} *satisfying condition (2.1.5) and the additional condition* $\epsilon < g_j$ *on* $\partial\Omega \times \{T\}$, *then*

$$||c_2 - c_1||_{\lambda}(\Omega) + ||u_2 - u_1||_{2+\lambda, 1+\frac{\lambda}{2}}(Q) \leq C(||g_2 - g_1||_{2+\lambda, 1+\frac{\lambda}{2}}(\partial\Omega \times (0,T)) + ||u_{T2} - u_{T1}||_{2+\lambda}(\Omega)), \tag{2.1.6}$$

where C *depends on the same parameters as in theorem (2.1.1), on* ϵ, *and on the norms of* g_j, u_{Tj} *used on the right side of inequality (2.1.6).*

(iii)(Existence) If in addition $u_T \in C^{2+\lambda}(\bar{\Omega})$ *satisfies the conditions*

$$-\Delta u_T + \frac{\partial v(,T)}{\partial t} \leq 0,\ u_T \geq 0 \text{ on } \Omega,\ g = u_T \text{ on } \partial\Omega \times \{T\}, \tag{2.1.7}$$

where v *is a solution to the initial boundary value problems (2.0.1)-(2.0.3) with* $c = 0$, *then there is a solution* $(u, c) \in C^{2+\lambda,\ 1+\frac{\lambda}{2}}(\bar{Q}) \times C^{\lambda}(\bar{\Omega})$ *to the inverse problem (2.0.1)-(2.0.3), (2.0.4).*

Proof. From our condition on the data of the parabolic equation (2.0.1)-(2.0.3), (2.0.4) we get the following problem :

$$\frac{\partial u}{\partial t} - \Delta u + cu = 0 \quad in \quad Q = \Omega \times (0,T),$$

$$u = 0 \quad on \quad \Omega \times \{0\},$$
$$u = g \quad on \quad \partial\Omega \times (0,T),$$
$$u = u_T \quad on \quad \Omega \times \{T\}. \tag{2.1.8}$$

We need to prove uniqueness, existence, and stability of equation (2.1.8).

(i) Using a possible translation with respect to t, one assumes that g is not zero on any set $\partial\Omega \times (0,\tau)$.

Now, since $f = 0$, $g \geq 0$ on $\partial\Omega \times (0,\tau)$, $u_0 = 0$, then $u > 0$ on $Q = \Omega \times (0,T)$ according to the positivity principles for the parabolic problems.

Differentiating equation (2.1.8) with respect to t we get

$$\frac{\partial^2 u}{\partial t^2} - \Delta\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial t} = 0 \quad in \quad Q = \Omega \times (0,T),$$

$$\frac{\partial u}{\partial t} = 0 \quad on \quad \Omega \times \{0\},$$

$$\frac{\partial u}{\partial t} = \frac{\partial g}{\partial t} \quad on \quad \partial\Omega \times (0,T),$$

$$\frac{\partial u}{\partial t} = \frac{\partial u_T}{\partial t} \quad on \quad \Omega \times \{T\}.$$

By applying the positivity principles to the previous equation and using that $f = 0$ on Q, $\frac{\partial g}{\partial t} \geq 0 \; on \; \partial\Omega \times (0,T)$, $\frac{\partial u_0}{\partial t} = 0$, then $\frac{\partial u}{\partial t} > 0 \; on \; Q$.

Now, if (u_1, c_1) and (u_2, c_2) are two solutions to the inverse parabolic problem (2.1.8), then subtracting the equations for them we get

$$\frac{\partial(u_2 - u_1)}{\partial t} - \Delta(u_2 - u_1) + (c_2 u_2 - c_1 u_1) = 0 \quad on \quad Q = \Omega \times (0,T),$$

$$u_2 - u_1 = 0 \quad on \quad \Omega \times \{0\},$$

$$u_2 - u_1 = g_2 - g_1 \quad on \quad \partial\Omega \times [0,T],$$
$$u_2 - u_1 = u_{T2} - u_{T1} \quad on \quad \Omega \times \{T\}.$$

Letting $u = u_2 - u_1$, we get

$$\frac{\partial u}{\partial t} - \Delta u + c_2 u_2 - c_1 u_1 = 0 \quad on \quad Q = \Omega \times (0,T).$$

Adding and subtracting $c_1 u_2$ we have

$$\frac{\partial u}{\partial t} - \Delta u + (c_2 - c_1)u_2 + c_1(u_2 - u_1) = 0 \quad on \quad Q.$$

Now, let $\alpha = u_2$ and $F = c_1 - c_2$ we get

$$\frac{\partial u}{\partial t} - \Delta u + \alpha F + c_1 u = 0 \quad on \quad Q.$$

So, we obtain the inverse source problem

$$\alpha F = -\frac{\partial u}{\partial t} + \Delta u - c_1 u \quad on \quad Q. \tag{2.1.9}$$

This is the same as the one mentioned in theorem(2.1.1), where the condition (2.1.1) is satisfied due to

$$\frac{\partial u}{\partial t},\ u > 0 \ on \ Q.$$

Hence, according to theorem (2.1.1) we conclude that $u = 0$ and $F = 0$. So, $u_2 = u_1$ and $c_2 = c_1$.

(ii) Let (u_1, c_1), (u_2, c_2) be two solutions of the inverse problem (2.1.8) with the data $g_1,\ u_{01},\ g_2,\ u_{02}$ satisfying the conditions

$$g_1,\ g_2 \geq 0,$$
$$\frac{\partial g_1}{\partial t},\ \frac{\partial g_2}{\partial t} \geq 0 \ (and\ not\ identically\ zero)\ on\ \partial\Omega \times (0,T),$$

and the additional condition $\epsilon < g_1,\ g_2\ on\ \partial\Omega \times \{T\}$.

By repeating the same subtraction procedure used in the proof of the uniqueness above and by letting $u = u_2 - u_1$, $\alpha = u_2$ and $F = c_1 - c_2$, we get the equation

$$\alpha F = -\frac{\partial u}{\partial t} + \Delta u - c_1 u.$$

Now, applying the estimate (2.1.2) in theorem (2.1.1) with $u = u_2 - u_1$ and $F = c_1 - c_2$ we get

$$||c_2 - c_1||_\lambda\,(\Omega) + ||u_2 - u_1||_{2+\lambda,1+\frac{\lambda}{2}}\,(Q) \leq C(||g_2 - g_1||_{2+\lambda,1+\frac{\lambda}{2}}\,(\partial\Omega\times(0,T)) + ||u_{T2} - u_{T1}||_{2+\lambda}\,(\Omega)).$$

(iii) The proof of existence can be found in [1]. $\square$

If, we are looking for principal coefficient a_0 of the problem (2.0.1)-(2.0.3), (2.0.4) with $a = 1$, $b = 0$, $c = 0$, $f = 0$, and $u_0 = 0$ then a similar result is valid.

2.2 Lateral overdetermination:single measurements

Inverse problems with final overdetermination are stable and computationally feasible, but they don't reflect interesting applied situations when one is given lateral data.[1]
This section is devoted to identification problems when one is given single boundary measurements; i.e, we are given one set of lateral boundary data $\{g, h\}$ on the lateral boundary $\partial\Omega \times (0, T)$ or on a part of it.
An uniqueness result is given by the next theorem, where the inverse problem is considered in the one-dimensional case .

Theorem 2.2.1. [1] *Let $\Gamma = \partial\Omega$, where $\Omega = (0, 1)$ in $\mathbb{R}$. Let $g(t, j)$, $j = 0, 1$, be the transform of a function $g^*(\tau, j) \in C^k([0, \infty))$ whose absolute value is bounded by $C\exp(C\tau)$ with some C and that satisfies the condition $g^{*(k-1)}(0) \neq 0$. Then the coefficient $c = c(x) \in L^\infty(\Omega)$ of equation (2.0.1) with $a_0 = a = 1$, $b = 0$ or the coefficient $a = a(x) \in C^2(\bar{\Omega})$ of the same equation with $a_0 = 1$, $b = 0$, $c = 0$ is uniquely determined by the Neumann data (2.0.5) of the parabolic problem (2.0.1)-(2.0.3) with $f = 0$, $u_0 = 0$, and $g = g(, j)$ at $x = j$.*

In the multidimensional case the initial condition (2.0.2) is replaced by the condition $u = u_0$ *on* $\Omega \times \{0\}$ where u_0 is positive .
Let P be a half-space in $\mathbb{R}^n$, e the exterior unit normal to ∂P, $\Gamma = \partial\Omega \bigcap P$, and x_0 a point of P such that $x^0.e \geq x.e$ when $x \in \Gamma$.

Theorem 2.2.2. [1] *let* $a = 1$*,* $b = 0$*,* $c = 0$*,* $f = 0$*, and* $g,\ u_0 \in C^l(\overline{\Omega}),\ \partial\Omega \in C^l$ *where* $(n+7)\backslash 2 \leq l$*. Assume that*

$$0 < \epsilon_0 < \triangle u_0 \quad on \quad \Omega_0 = \Omega \bigcap P. \tag{2.2.1}$$

Then the coefficient $a_0 = a_0(x) \in C^l(\overline{\Omega_0})$ *satisfying the condition*

$$\nabla a_0.e \leq 0,\ 0 \leq a_0 + \frac{1}{2}\nabla a_0.(x - x_0) \quad on \quad \Omega_0,$$

is uniquely determined on Ω_0 *by the additional Neumann data (2.0.5).*

In the following theorem, the uniqueness result of the coefficient $c = c(x)$ is given where the condition (2.2.1) in theorem (2.2.2) is replaced by the condition $0 < \epsilon_0 < u_0 \ \ on \ \ \Omega_0$.

Theorem 2.2.3. [1] *Let* $a_0 = a = 1$*,* $b = 0$*,* $f = 0$ *and* $g,\ u_0,\ c \in C^l$ *where* $(n+7)\backslash 2 \leq l$*. Let the initial data satisfy the condition*

$$0 < \epsilon \leq u_0 \quad on \quad \Omega_0 = \Omega \bigcap P.$$

Then the coefficient $c = c(x)$ *is uniquely determined on* Ω_0 *by the additional Neumann data (2.0.5).*

2.3 Lateral overdetermination:many measurements

This section is devoted to identification problem when one is given the Neumann data for all (regular) Dirichlet data (2.0.3). In other words we know the lateral Dirichlet-to- Neumann map $\bigwedge_L : g \longrightarrow h$.
We start this section by answering the uniqueness question concerning the coefficient $a = a(x)$ and $c = c(x)$ of the problem (2.0.1)-(2.0.3), (2.0.5).
A proof of the next theorem can be done using the stabilization of solutions of parabolic problems when $t \longrightarrow \infty$ and by reducing our inverse parabolic problem to inverse elliptic problem with parameter where one can use results concerning elliptic equations.

Theorem 2.3.1. [1] *Let* $a_0 = 1$*,* $b = 0$*, and let* a *be a scalar matrix. Let* $f = 0$ *and* $u_0 = 0$*. Let* $\Gamma = \partial\Omega$*. Then the lateral Dirichlet-to-Neumann map* $\bigwedge_L$ *uniquely determines* $a \in H_2^\infty(\Omega)$ *and* $c \in L^\infty(\Omega)$*.*

Proof. The substitution $u = v\exp(\lambda t)$ transforms the equation

$$\frac{\partial u}{\partial t} - div(a\nabla u) + cu = 0 \ \ in \ \ Q,$$

into the following equation

$$\frac{\partial v}{\partial t} - div(a\nabla v) + (c+\lambda)v = 0 \ \ in \ \ Q. \tag{2.3.1}$$

Choosing λ large we achieve that $c + \lambda \geq 0$.

Let $g = g^0\phi$ where $g^0 \in C^2(\partial\Omega)$ and $\phi \in C^2(\partial\Omega)$ satisfies the conditions

$$\phi = 0 \ \ on \ \ (-\infty, \frac{T}{4}),$$

$$\phi = 1 \ \ on \ \ (\frac{3T}{4}, \infty).$$

Extend g onto $\partial\Omega \times (T, \infty)$ as g^0.

According to theorem (2.0.3), the solution $v(x,t)$ is analytic with respect to t where $\frac{3T}{4} \leq t$ since the coefficients of the equation (2.3.1) and the lateral boundary condition g do not depend on t when $t \geq \frac{3T}{4}$.

Equation (2.3.1) satisfies all the conditions of the maximum principle and it's coefficients and the lateral boundary conditions don't depend on $t > \frac{3T}{4}$. Hence, theorem (2.0.4) guarantees that $||v(,t) - v^0||_\infty (\Omega) \leq C\exp(-t\theta)$.

Choosing t large enough, then we get

$$\lim_{t\longrightarrow\infty} \left|\left|v(,t) - v^0\right|\right|_\infty (\Omega) = 0, \tag{2.3.2}$$

where v^0 is an unique solution to the Dirichlet problem

$$-div(a\nabla v^0) + (c+\lambda)v^0 = 0 \ \ in \ \ \Omega,$$

$$v^0 = g^0 \ \ on \ \ \ \partial\Omega. \tag{2.3.3}$$

Theorems and results concerning inverse elliptic problems can be applied to equation (2.3.3).

The maximum principle for parabolic equations yields $||v||_{\infty}(\Omega \times \mathbb{R}^+) \leq C$, so the local L^p-estimates for parabolic boundary value problems according to theorem (2.0.5) gives $||v||_{2,1;p}(\Omega \times (s-1, s+1)) \leq C$.
Differentiating equation (2.3.3) with respect to t and again using the local estimates, we derive from the previous bound that $\left|\left|\frac{\partial v}{\partial t}\right|\right|_{2,1;p}(\Omega \times (s, s+1)) \leq C$.

Trace theorems give $||v(,t)||_{(2)}(\Omega) \leq C$, from this estimate and estimate (2.3.2) and by using interpolation theorems it follows that

$$\lim_{t \longrightarrow \infty} \left|\left|v(,t) - v^0\right|\right|_{(1)}(\Omega) = 0.$$

Again by trace theorems, we obtain that $\lim_{t \longrightarrow \infty} \frac{\partial v(,t)}{\partial \nu} = \frac{\partial v^0}{\partial \nu}$ in $H_{-\frac{1}{2}}(\partial\Omega)$.

Since $a\frac{\partial v(,t)}{\partial \nu}$ on $\partial\Omega$ is given, we conclude that $a\frac{\partial v^0}{\partial \nu}$ on $\partial\Omega$ is given as well. Thus, the Dirichlet-to-Neumann map for the elliptic equation (2.3.3) is obtained for all $\lambda \geq -c$. Hence, a and ∇a on $\partial\Omega$ are uniquely determined according to theorem (5.1.1) in [1].

The Riccati substitution $v^0 = a^{-\frac{1}{2}}w$ transforms the equation (2.3.3) into the Schrodinger equation

$$-\Delta w + a^{-\frac{1}{2}}(\Delta a^{\frac{1}{2}} + c + \lambda)w = 0 \quad in \quad \Omega.$$

By theorems (5.3.1) and (5.4.1) in [1] the coefficient $c^* = a^{-\frac{1}{2}}(\Delta a^{\frac{1}{2}} + c + \lambda)$ of w (for any λ) is uniquely determined by it's Dirichlet-to-Neumann map. Hence, $a^{-\frac{1}{2}}$ is uniquely determined as the coefficient of λ in c^*, therefore c is uniquely determined as well.

□

Theorem 2.3.2. [1] *Let $a = 1$ and $n \geq 3$. There is positive ϵ such that if $||curlb||_{\infty} < \epsilon$, then the coefficients a_0, $c \in L^{\infty}(\Omega)$, $b \in H_2^{\infty}(\Omega)$ are uniquely determined by Λ_l.*

Proof. The substitution $u = v\exp(\lambda t)$ transforms the equation

$$a_0\frac{\partial u}{\partial t} - \triangle u + b.\nabla u + cu = 0 \quad in \quad \Omega,$$

into the following equation

$$a_0\frac{\partial v}{\partial t} - \triangle v + b.\nabla v + (c + a_0\lambda)v = 0 \quad in \quad \Omega.$$

As in the proof of theorem (2.3.1), Λ_l uniquely determines the Dirichlet-to-Neumann map Λ for the elliptic equation

$$-\Delta v^0 + b.\nabla v^0 + (c + a_0\lambda)v^0 = 0 \;\; in \;\; \Omega.$$

From the recent results of [1] and of [19] we conclude that $curlb$, $c + a_0\lambda$ are uniquely identified by Λ for all λ. So $curlb$, c, a_0 are unique. □

Theorem 2.3.3. [1] *let $a_0 = 1$, $b = 0$, $a = 1$. Let P be a half-space, $n \geq 2$. Define $\Omega_0 = \Omega \bigcap P$, $\Gamma = \partial\Omega \bigcap P$. Then the local Dirichlet-to-Neumann map $\Lambda_{l,\Gamma} : g \longrightarrow \partial_\nu u$ on $\Gamma \times (0,T)$, $suppg \subset \Gamma \times (0,T)$ uniquely determines $c \in L^\infty(\Omega)$ on Ω_0.*

2.4 Interior sources

All the results presented until now concerning identification of coefficients have been obtained when the source term $f = 0$.

In this section we assume that the source term f is the Dirac delta function $\delta(-y,-s)$ with the pole at a point (y,s) and that the Dirichlet boundary data $g = 0$ in our problem (2.0.1)-(2.0.3). Let Γ be any open part of $\partial\Omega$. Fix a function g^* satisfying the following conditions:

$$g^* \geq 0 \; on \; \Gamma \times (0,T), g^* \; is \; not \; identically \; zero \; on \; \Gamma \times (s_0,T),$$
$$g^* = 0 \; on \; (\partial\Omega\backslash\Gamma) \times (0,T), g^* \in C^2(\partial\Omega \times [0,T]),$$
$$g^* = \frac{\partial g^*}{\partial t} = 0 \; on \; \partial\Omega \times \{T\}. \tag{2.4.1}$$

We are looking for the coefficient $c = c(x) \in C^\lambda(\bar{Q})$ of the parabolic initial boundary value problem (2.0.1)-(2.0.3) with $a_0 = 1$, $a = 1$, $b = 0$ subject to the Dirichlet boundary data $g = 0$.

The solution of this parabolic boundary value problem is denoted by $u(x,t;y,s)$. It is Green's function of the first boundary value problem for our equation.

Theorem 2.4.1. [1] *The Neumann data (2.0.5) of u given for all (y,s) in a neighborhood V of a point $(y_0,s_0) \in Q$ uniquely (and in a stable way) determines $c(y_0,s_0)$.*

Proof. Let v be a solution to the backward heat equation

$$\frac{\partial v}{\partial t} + \Delta v = 0 \ in \ Q, \tag{2.4.2}$$

with initial and lateral boundary conditions

$$v = 0 \ on \ \Omega \times \{T\},$$

$$v = g^* \ on \ \partial\Omega \times (0, T), \tag{2.4.3}$$

where g^* satisfy condition (2.4.1).
Using the known definition of a generalized solution to our parabolic boundary value problem (2.0.1)-(2.0.3), (2.0.5) with the test function v, we have

$$\int_Q u(-\frac{\partial v}{\partial t} - \Delta v + cv) - \int_{\Gamma\times(0,T)} g^* \frac{\partial u}{\partial \nu} = v(y, s).$$

Using equation (2.4.2) we get

$$\int_Q u(cv) - \int_{\Gamma\times(0,T)} g^* \frac{\partial u}{\partial \nu} = v(y, s).$$

Now the function

$$U(y, s) = \int_Q cu(; y, s)v = w(y, s) + v(y, s),$$

is given when $(y, s) \in V$ since $v(y, s)$ is the solution to (2.4.2) and $w(y, s) = \int_{\Gamma\times(0,T)} g^* \frac{\partial u}{\partial \nu}$ is known according to the Neumann condition (2.0.5).

Since $u(x, t; y, s)$ is the Green's function for the equation $\frac{\partial u}{\partial t} - \Delta u + cu = f$, we have

$$-\frac{\partial U}{\partial s} - \Delta_y U + cU = cv.$$

Now, since $U(y, s) = w(y, s) + v(y, s)$, then

$$-\frac{\partial U}{\partial s} - \Delta_y U + cU = -\frac{\partial w}{\partial s} - \Delta_y w + cw - \frac{\partial v}{\partial s} - \Delta_y v + cv.$$

Using equation (2.4.2), we conclude that

$$-\frac{\partial w}{\partial s} - \Delta_y w + cw = 0 \ in \ Q. \tag{2.4.4}$$

According to conditions (2.4.3) the function U is both 0 on $\Omega \times \{T\}$ and on $\partial\Omega \times (0,T)$, hence we conclude that

$$w = U - v = 0 \ on \ \Omega \times \{T\},$$

$$w = -g^* \ on \ \partial\Omega \times (0,T).$$

Due to condition (2.4.1) we have $U = 0 \leq v \ on \ \partial\Omega \times (0,T)$ and v is not identically zero there, so $U < v \ on \ \partial\Omega \times (0,T)$.

By the positivity principle for parabolic equations, we conclude that $w < 0 \ in \ V$. Hence, we can divide by $w = U - v \ in \ V$ to find $c(y,s)$ from the differential equation (2.4.4) as follows

$$c(y,s) = \frac{(\frac{\partial w}{\partial s} + \Delta w)}{w(y,s)}, \quad (y,s) \in V.$$

□

Consider the problem (2.0.1), (2.0.2), (2.0.3) with $a = 1$, $b = 0$, $c = 0$, unknown positive a_0, $\frac{\partial a_0}{\partial t} \in C^\lambda(\bar{Q})$, boundary data $g = 0$, zero initial data, and the source term $f = \delta(-y,-s)$. Let $u(;y,s)$ be a solution of this problem.

Theorem 2.4.2. [1] *The Neumann data (2.0.5) for (y,s) in a neighborhood of (y_0,s_0), $0 < s_0 < T$ uniquely determine $a_0(y_0,s_0)$.*

Proof. Let v be defined as in the proof of theorem (2.4.1), with the condition (2.4.1) replaced by

$$\frac{\partial g^*}{\partial t} \geq 0 \quad on \quad \Gamma \times (0,T),$$

$$g^* \neq 0 \quad on \quad any \ \Gamma \times (0, T-\tau),$$

$$g^* = 0 \quad on \quad (\partial\Omega \backslash \Gamma) \times (0,T), \quad g^* \in C_0^3(\Gamma \times (0,T)). \tag{2.4.5}$$

By the definition of a generalized solution to our parabolic boundary value problem with $f = \delta(-y,-s)$ we have

$$v(y,s) = \int_Q u(-a_0 \frac{\partial v}{\partial t} - \triangle v) - \int_{\Gamma \times (0,T)} g^* \frac{\partial u}{\partial \nu}.$$

Using equation (2.4.2), put $\Delta v = -\frac{\partial v}{\partial t}$, hence we obtain

$$v(y,s) = \int_Q u(-a_0 + 1)\frac{\partial v}{\partial t} - \int_{\Gamma \times (0,T)} g^* \frac{\partial u}{\partial \nu}.$$

The function

$$U(y,s) = \int_Q (1-a_0)u(;y,s)\frac{\partial v}{\partial t}dxdt = v(y,s) + \int_{\Gamma\times(0,T)} g^*\frac{\partial u}{\partial \nu}, \qquad (2.4.6)$$

is a known function when (y,s) is close to (y_0,s_0) since both $v(y,s)$ and the integral $\int_{\Gamma\times(0,T)} v\frac{\partial u}{\partial \nu}$ are known according to equation (2.4.2) and the Neumann condition (2.0.5).

Since $u(x,t;y,s)$ is the Green's function for the equation $a_0\frac{\partial u}{\partial t} - \Delta u = f$ we have from (2.4.6)

$$-a_0(y)\frac{\partial U}{\partial s} - \triangle_y U = (1-a_0)\frac{\partial v}{\partial s} \quad in \quad Q,$$

$$U = 0 \quad on \quad \partial Q \bigcap \{t>0\}. \qquad (2.4.7)$$

From (2.4.2) it follows that

$$-a_0\frac{\partial v}{\partial s} - \triangle v = (1-a_0)\frac{\partial v}{\partial s} \quad in \quad Q,$$

$$v = 0 \quad on \quad \Omega\times\{T\}. \qquad (2.4.8)$$

Subtract the two equations (2.4.7) and (2.4.8) we get

$$-a_0\frac{\partial(v-U)}{\partial s} - \Delta(v-U) = 0 \; in \; Q,$$

$$v - U = 0 \; on \; \partial\Omega\times\{T\}. \qquad (2.4.9)$$

So $\frac{\partial(v-U)}{\partial s}$ solves a homogeneous backward parabolic equation in Q and is zero on $\Omega\times\{T\}$.
From (2.4.5) we conclude that $\frac{\partial(v-U)}{\partial s}$ has non-negative lateral Dirichlet data $\frac{\partial g^*}{\partial s}$, so by positivity principles $\frac{\partial(v-U)}{\partial s} > 0$ in Q and we can find $a_0(y)$ from (2.4.7) as follows

$$a_0(y_0) = \frac{(\frac{\partial v}{\partial s} + \triangle_y U)}{(\frac{\partial(v-U)}{\partial s})(y_0,s_0)}.$$

□

Chapter 3

Solution Strategies

In this chapter we will review some numerical techniques for solving inverse parabolic problems. In solving them we face two difficulties: ill-posedness and nonlinearity.
In most practical numerical procedures nonlinearity is removed by replacing the original problem by its linearization around constant coefficient while ill-posedness must be handled either by using some additional priori knowledge about the problem which stabilize the problem or by using appropriate numerical methods called regularization techniques.[1]

3.1 Numerical solution for the inverse heat problem in R

Consider the heat equation in one dimensional case:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad for\ all\ x\ in\ \mathbb{R},\ t > 0, \tag{3.1.1}$$

with initial values given by

$$u(x,0) = f(x), \quad for\ all\ x\ in\ \mathbb{R}.$$

It's well known that if $f(x)$ is continuous and bounded for $x \in \mathbb{R}$, then

$$u(x,t) = \int_{-\infty}^{\infty} K(x,y,t)f(y)dy = (4\pi t)^{\frac{-1}{2}} \int_{-\infty}^{\infty} \exp(\frac{-(x-y)^2}{4t})f(y)dy. \tag{3.1.2}$$

The inverse heat problem in $\mathbb{R}$ is the problem of recovering the initial data $f(y)$ for all $y \in \mathbb{R}$ when for some $t > 0$, $u(x,t)$ is given for all $x \in \mathbb{R}$.

Numerical solution of this inverse heat problem consists of approximating the initial data using sinc expansion and the domain using discrete time and spatial sampling, then the problem is reduced to Toeplitz linear system that can be solved based on the preconditioned conjugate gradient method.

An approximation of $f(y)$ in (3.1.2) is given by

$$f(y) \approx \sum_{k=-\infty}^{\infty} f(kh) sinc(\frac{y-kh}{h}).$$

Then (3.1.2) becomes

$$u(x,t) \approx \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} f(kh) \int_{-\infty}^{\infty} \exp(\frac{-(x-y)^2}{4t}) sinc(\frac{y-kh}{h})dy.$$

By letting $x = x_j = jh$ and $t_0 = \frac{h}{2\pi}$, and after some simplification we finally have

$$u(x_j,t_0) \approx \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f(kh) \int_{-\pi}^{\pi} \exp(\frac{-\tau^2}{4\pi^2}) \exp(i(k-j)\tau)d\tau,$$

$$\equiv \sum_{k=-\infty}^{\infty} f(kh)\beta_{k-j},$$

where

$$\beta_{k-j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(\frac{-\tau^2}{4\pi^2}) \exp(i(k-j)\tau)d\tau,$$

are the Fourier coefficients of the function

$$g(\tau) = \exp(-\frac{\tau^2}{4\pi^2}).$$

For fixed $n > 0$, we then have the discrete system:

$$u(x_j, t_0) = \sum_{k=-n}^{n} f(kh)\beta_{k-j}.$$

Or in matrix form:

$$B_{2n+1}\vec{f} = \vec{u},$$

where $B_{2n+1}$is a symmetric Toeplitz matrix given by:

$$B_{2n+1} = \begin{pmatrix} \beta_0 & \beta_1 & \beta_{2n} \\ \beta_{-1} & \beta_0 & \beta_{2n-1} \\ \beta_{-2n} & \beta_{-2n+1} & \beta_0 \end{pmatrix},$$

and $\vec{f}$, $\vec{u}$ are two vectors that are given by

$\vec{f} = (f(-nh), f(-nh+h), ..., f(nh-h), f(nh))^t$,
and
$\vec{u} = (u(x_{-n}, t_0), ..., u(x_0, t_0), ..., u(x_n, t_0))^t$.

The inverse heat problem (3.1.1) has been converted into a problem of solving the Toeplitz system. This system can be solved using iterative method such as the conjugate gradient method.

By finding the inverse of B_{2n+1}we can approximate $f(y)$ given $\vec{u}$.

3.2 Solving the one dimensional inverse parabolic problem using Chebyshev polynomials of the first kind

In this section a numerical technique for the inverse problem of finding an unknown boundary condition from an additional information is presented.

Let us consider the following inverse heat conduction problem :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad 0 < t < T_0, \tag{3.2.1}$$

$$u(x, 0) = f(x), \quad 0 < x_1 < x < 1, \tag{3.2.2}$$

$$u(x_1,t) = g(t), \quad 0 < t < T_0, \tag{3.2.3}$$

$$u_x(1,t) = h(t), \quad 0 < t < T_0, \tag{3.2.4}$$

where T_0 is a given positive constant, $f(x)$ is piecewise-continuous known function, $g(t)$ and $h(t)$ are infinitely differentiable functions, and the sensor is located at $x_1,\ 0 < x_1 < 1$.

In this problem we need to determine the temperature $u(x,t)$ and heat flux $u_x(0,t)$. This can be done by dividing the problem (3.2.1)-(3.2.4) into two separate problems.

The first problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x_1 < x < 1, \quad 0 < t < T_0,$$

$$u(x,0) = f(x), \quad x_1 < x < 1,$$

$$u(x_1,t) = g(t), \quad 0 < t < T_0,$$

$$u_x(1,t) = h(t), \quad 0 < t < T_0.$$

This problem is a direct problem with known boundary conditions and it has a unique stable solution.

The second problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < x_1, \quad 0 < t < T_0, \tag{3.2.5}$$

$$u(x_1,t) = g(t), \quad 0 < t < T_0. \tag{3.2.6}$$

To obtain the heat in the body and heat flux at the boundary $x = 0$ we need an extra condition:

$$u_x(x_1,t) = k(t), \quad 0 \leq t \leq T_0, \tag{3.2.7}$$

where $k(t)$ can be obtained from the solution of the first direct problem.

A stable solution for the problem (3.2.5)-(3.2.7) can be obtained by using the transformations

$$\tau = -1 + \frac{2}{T_0}t,$$

and

$$T(x,\tau) = u(x, \frac{(1+\tau)T_0}{2}).$$

Then(3.2.5)-(3.2.7)becomes

$$\frac{\partial T}{\partial \tau} = \frac{T_0}{2}\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < x_1, \quad -1 < \tau < 1, \tag{3.2.8}$$

$$T(x_1,\tau) = G(\tau), \quad -1 \le \tau \le 1, \tag{3.2.9}$$

$$\frac{\partial T(x_1,\tau)}{\partial x} = K(\tau), \quad -1 \le \tau \le 1. \tag{3.2.10}$$

Now we seek a solution for this problem in the form

$$T(x,\tau) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij} T_j(\tau)(x-x_1)^i, \tag{3.2.11}$$

where $T_j(\tau)$ is a ChebyshevT polynomials of degree j, and C_{ij} are constant numbers which must be determined.

By putting (3.2.11) in (3.2.8) we get

$$\frac{T_0}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij} T_j(\tau) i(i-1)(x-x_1)^{i-2} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij} T_j^{'}(\tau)(x-x_1)^i,$$

then

$$\frac{T_0}{2(x-x_1)^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij}(T_j(\tau) i(i-1) - T_j^{'}(\tau))(x-x_1)^i = 0.$$

And, by equating the coefficients of each power of $(x-x_1)^i$ to zero, we find

$$C_{i+2,j} = \frac{2T_{j+1}^{'}(\tau)}{T_0(i+1)(i+2)T_j(\tau)} C_{i,j+1}.$$

From $T(x_1,\tau) = \sum_{j=0}^{\infty} C_{0,j} T_j(\tau) = G(\tau)$, we conclude that

$$C_{0,0} = \frac{1}{\pi}\int_{-1}^{1} \frac{G(\tau)}{\sqrt{1-\tau^2}} d\tau,$$

and

$$C_{0,j} = \frac{2}{\pi}\int_{-1}^{1} \frac{G(\tau)T_j(\tau)}{\sqrt{1-\tau^2}} d\tau.$$

Similarly from $\frac{\partial T(x_1,\tau)}{\partial x} = K(\tau)$ we obtain

$$C_{1,0} = \frac{1}{\pi}\int_{-1}^{1} \frac{K(\tau)}{\sqrt{1-\tau^2}}d\tau,$$

and

$$C_{1,j} = \frac{2}{\pi}\int_{-1}^{1} \frac{K(\tau)T_j(\tau)}{\sqrt{1-\tau^2}}d\tau.$$

By substituting the obtained results in (3.2.11), we find

$$T(x,\tau) = \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}(\frac{2}{T_0})^s(\prod_{k=1}^{s}\frac{T'_{j+k}(\tau)}{T_{j+k-1}(\tau)})(\frac{C_{0,j+s}}{(2s)!}T_j(\tau)(x-x_1)^{2s}+\frac{C_{1,j+s}}{(2s+1)!}T_j(\tau)(x-x_1)^{2s+1}). \tag{3.2.12}$$

Now, if $G(\tau)$ and $K(\tau)$ belong to Holmgren class $H(x_1, 1, C_1, 0)$ then (3.2.12) is a solution to the problem (3.2.8)-(3.2.10).

Theorem 3.2.1. *If $G(\tau)$ and $K(\tau)$, belong to Holmgren class $H(x_1, 1, C_1, 0)$ and $G(\tau)$ and $K(\tau)$, are approximated by two polynomials of degree n, then the approximate solution (3.2.12) is stable.*

The proof of this theorem can be found in [4].
Now, if $G(\tau)$ and $K(\tau)$ are approximated by polynomials of degree n, then by the orthogonality of ChebyshevT polynomials with respect to the weight function $(1-\tau^2)^{\frac{-1}{2}}$, and the following properties

$$\int_{-1}^{1} \frac{G(\tau)T_{j+s}(\tau)}{\sqrt{1-\tau^2}}d\tau = 0, \quad j+s > n,$$

and

$$\int_{-1}^{1} \frac{K(\tau)T_{j+s}(\tau)}{\sqrt{1-\tau^2}}d\tau = 0, \quad j+s > n,$$

we will obtain an approximate solution for the problem (3.2.8)-(3.2.10) in the form:

$$T(x,\tau) \cong \sum_{s=0}^{n}\sum_{j=0}^{n}(\frac{2}{T_0})^s(\prod_{k=1}^{s}\frac{T'_{j+k}(\tau)}{T_{j+k-1}(\tau)})(\frac{C_{0,j+s}}{(2s)!}T_j(\tau)(x-x_1)^{2s}+\frac{C_{1,j+s}}{(2s+1)!}T_j(\tau)(x-x_1)^{2s+1}).$$

The heat flux at $x = 0$ can be obtained by differentiating the formula (3.2.12) with respect to x and putting $x = 0$.

3.3 Numerical solution of an inverse diffusion problem based on the Laplace transform and the finite difference method

In this section an algorithm for numerical solving an inverse nonlinear diffusion problem is proposed. The numerical solution of inverse diffusion problem requires to determine an unknown diffusion coefficient from additional information.

Consider the following initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(a(u)\frac{\partial u}{\partial x}), \quad 0 < x < 1, \quad 0 < t < T, \tag{3.3.1}$$

$$u(x,0) = \phi(x), \quad 0 < x < 1, \tag{3.3.2}$$

$$-a(u(0,t))\frac{\partial u(0,t)}{\partial x} = g_0(t), \quad 0 < t < T, \tag{3.3.3}$$

$$-a(u(1,t))\frac{\partial u(1,t)}{\partial x} = g_1(t), \quad 0 < t < T, \tag{3.3.4}$$

$$u(0,t) = f(t), \quad 0 < t < T, \tag{3.3.5}$$

where $\varphi(x)$, $g_0(t)$, $g_1(t)$ and $f(t)$ are continuous known functions. Here we are looking for both the coefficient $a(u(x,t))$ and the solution $u(x,t)$ for the problem (3.3.1)-(3.3.5).

For an unknown function $a(u)$, an additional information (3.3.5) must be added to provide a unique solution $(u, a(u))$ to the inverse problem (3.3.1)-(3.3.5).

To get a solution of problem (3.3.1)-(3.3.5) we start by linearzing the nonlinear terms in equations (3.3.1), (3.3.3) and (3.3.4) using Taylor's series expansion. Therefore, we obtain

$$\frac{\partial}{\partial x}(a(u)\frac{\partial u}{\partial x}) = a(\bar{u})\frac{\partial^2 u}{\partial x^2},$$

$$-a(u(0,t))\frac{\partial u(0,t)}{\partial x} = -a(\bar{u}(0,t))\frac{\partial u(0,t)}{\partial x},$$

$$-a(u(1,t))\frac{\partial u(1,t)}{\partial x} = -a(\bar{u}(1,t))\frac{\partial u(1,t)}{\partial x},$$

where $\bar{u} = (\bar{u}_0, \bar{u}_1, ..., \bar{u}_N)$.
Using Laplace transform, we can remove the time dependent terms in the above equations, so we get:

$$a(\bar{u})\frac{\partial^2 \tilde{u}}{\partial x^2} = \mathcal{L}\{a(\bar{u})\frac{\partial^2 u}{\partial x^2}\} = \mathcal{L}\{\frac{\partial u}{\partial t}\} = s\mathcal{L}\{u\} - u(x,0) = s\tilde{u} - \phi(x), \ \ 0 < x < 1,$$

$$-a(\bar{u})\frac{\partial \tilde{u}}{\partial x} = \mathcal{L}\{-a(\bar{u})\frac{\partial u}{\partial x}\} = \mathcal{L}\{g_0(t)\} = \int_0^\infty \exp(-st) g_0(t)\, dt = G_0(s), \ \ x = 0,$$

$$-a(\bar{u})\frac{\partial \tilde{u}}{\partial x} = \mathcal{L}\{-a(\bar{u})\frac{\partial u}{\partial x}\} = \mathcal{L}\{g_1(t)\} = \int_0^\infty \exp(-st) g_1(t)\, dt = G_1(s), \ \ x = 1,$$

where $\tilde{u}$, $\frac{\partial \tilde{u}}{\partial x}$, $\frac{\partial^2 \tilde{u}}{\partial x^2}$, $G_0(s)$, and $G_1(s)$ are Laplace transform of u, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $g_0(t)$ and $g_1(t)$.

Now, we use central finite difference approximation for discretizing the following problem

$$a(\bar{u})\frac{\partial^2 \tilde{u}}{\partial x^2} = s\tilde{u} - \phi(x), \ \ 0 < x < 1, \tag{3.3.6}$$

$$-a(\bar{u})\frac{\partial \tilde{u}}{\partial x} = G_0(s), \ \ x = 0, \tag{3.3.7}$$

$$-a(\bar{u})\frac{\partial \tilde{u}}{\partial x} = G_1(s), \ \ x = 1. \tag{3.3.8}$$

Therefore, we obtain

$$a(\bar{u}_\mu)\frac{\tilde{u}_{\mu+1} - 2\tilde{u}_\mu + \tilde{u}_{\mu-1}}{h^2} = s\tilde{u}_\mu - \phi(\mu h), \ \ \mu = 0, 1, ..., N,$$

$$-a(\bar{u}_0)\frac{\tilde{u}_1 - \tilde{u}_{-1}}{2h} = G_0(s), \ \ x = 0,$$

$$-a(\bar{u}_N)\frac{\tilde{u}_{N+1} - \tilde{u}_{N-1}}{2h} = G_1(s), \ \ x = 1.$$

Then the previous problem can be written in matrix form as follows:

$$A\tilde{U} = B.$$

To find $\tilde{U}$, we can use Gaussian elimination method, then using inversion of the Laplace transform we get $U^t = (u_0\ u_1\ ...\ u_N)$.

The unknown function $a(u)$ approximated as

$$a(u) = a_0 + a_1 u + a_2 u^2 + ... + a_q u^q,$$

where a_0, a_1, ..., a_q are constant which remain to be determined simultaneously.

3.4 A tau method for solving the one-dimensional parabolic inverse problem based on the shifted Legendre polynomials

A direct computational technique is presented for the inverse problem of finding a source parameter $p(t)$ in the following diffusion equation:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + p(t)w + q(x,t), \quad 0 \le x \le l, \quad 0 \le t \le \tau, \tag{3.4.1}$$

with initial condition

$$w(x,0) = f(x), \quad 0 \le x \le l, \tag{3.4.2}$$

and boundary conditions

$$w(0,t) = g_0(t), \quad 0 < t \le \tau, \tag{3.4.3}$$

$$w(l,t) = g_1(t), \quad 0 < t \le \tau, \tag{3.4.4}$$

subject to the overspecification at a point in the spatial domain

$$w(x_0,t) = K(t), \quad 0 \le t \le \tau, \tag{3.4.5}$$

where f, g_0, g_1, q and K are known functions, while the functions w and p are unknown.

Starting with employing a pair of transformations

$$r(t) = \exp(-\int_0^t p(s)ds), \tag{3.4.6}$$

$$u(x,t) = r(t)w(x,t), \tag{3.4.7}$$

to the problem (3.4.1)-(3.4.5), we get

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + r(t)q(x,t), \quad 0 \le x \le l, \quad 0 \le t \le \tau, \tag{3.4.8}$$

subject to

$$u(x,0) = f(x), \quad 0 \le x \le l, \tag{3.4.9}$$

$$u(0,t) = r(t)g_0(t), \quad 0 < t \le \tau, \tag{3.4.10}$$

$$u(l,t) = r(t)g_1(t), \quad 0 < t \le \tau, \tag{3.4.11}$$

and

$$u(x_0,t) = r(t)K(t), \quad 0 \le t \le \tau. \tag{3.4.12}$$

Integrating equation (3.4.8) from 0 to t and using equation (3.4.9) we have

$$u(x,t) - f(x) = \int_0^t \frac{\partial^2 u}{\partial x^2}(x,t^{'})dt^{'} + \int_0^t r(t^{'})q(x,t^{'})dt^{'}. \tag{3.4.13}$$

A function $q(x,t)$ of two independent variables defined for $0 \le x \le l$ and $0 \le t \le \tau$ can be expanded in terms of double-shifted Legendre polynomials as

$$q(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} q_{ij}P_i^{\tau}(t)P_j^{l}(x) = \psi^T(t)\mathbf{Q}\phi(x), \tag{3.4.14}$$

where $\mathbf{Q}$ is a $(n+1)\times(m+1)$ known matrix.
The function $r(t)$ can be expanded in terms of $n+1$ shifted Legendre series as

$$r(t) = \sum_{k=0}^{n} b_k P_k^{\tau}(t) = \mathbf{B}^T\psi(t), \tag{3.4.15}$$

where $\mathbf{B}^T = [b_0, b_1, ..., b_n]^T$ is an unknown vector.

Using equation (0.4.1), we have

$$u_{xx}(x,t) = \psi^T(t)\mathbf{A}\frac{d^2\phi(x)}{dx^2},$$

then

$$\int_0^t u_{xx}(x,t^{'})dt^{'} = (\int_0^t \psi^T(t^{'})dt^{'})\mathbf{A}(\frac{d^2\phi(x)}{dx^2}) = (\mathbf{P}\psi(t))^T\mathbf{A}\mathbf{D}^2\phi(x) = \psi^T(t)\mathbf{P}^T\mathbf{A}\mathbf{D}^2\phi(x), \tag{3.4.16}$$

where $\mathbf{P}$ is an $(n+1)\times(n+1)$ operational matrix of integration and $\mathbf{D}$ is the $(m+1)\times(m+1)$ operational matrix of derivative.
And by using equation(0.4.1),(3.4.14), and (3.4.15) we have

$$\int_0^t r(t^{'})q(x,t^{'})dt^{'} = (\int_0^t \mathbf{B}^T\psi(t^{'})\psi^T(t^{'})dt^{'})\mathbf{Q}\phi(x). \tag{3.4.17}$$

Now suppose

$$w_{k,i,j} = \int_0^\tau P_k^\tau(t)P_j^\tau(t)P_i^\tau(t)dt, \quad k,j,i = 0,1,...,n.$$

Then we see that

$$\mathbf{B}^T\psi(t)\psi^T(t) = \psi^T(t)\mathbf{H},$$

where $\mathbf{H}$ is an $(n+1)\times(n+1)$ matrix given as $\mathbf{H}_{ij} = (\frac{2i+1}{\tau})\sum_{k=0}^n b_k w_{k,j,i},\ \ i,j = 0,1,...n$.

Using the previous results equation (3.4.17) can be written as

$$\int_0^t r(t^{'})q(x,t^{'})dt^{'} = \psi^T(t)\mathbf{P}^T\mathbf{H}\mathbf{Q}\phi(x). \tag{3.4.18}$$

Expanding $f(x)$ by $(m+1)$ terms of shifted Legendre series we get

$$f(x) = \sum_{k=0}^m f_k P_k^l(x) = \psi^T(t)\mathbf{F}\phi(x), \tag{3.4.19}$$

where $\mathbf{F}$ is a known $(n+1)\times(m+1)$ matrix.

Substituting equations (0.4.1), (3.4.16), (3.4.18), and (3.4.19) into equation (3.4.13) we get

$$\psi^T(t)\mathbf{A}\phi(x) - \psi^T(t)\mathbf{F}\phi(x) = \psi^T(t)\mathbf{P}^T\mathbf{H}\mathbf{Q}\phi(x) + \psi^T(t)\mathbf{P}^T\mathbf{A}\mathbf{D}^2\phi(x).$$

3.5 A high-order compact finite difference method for solving an inverse problem of the one-dimensional parabolic equation

In this section a fourth-order efficient numerical method is used to calculate the function $u(x,t)$ and the unknown coefficient $a(t)$ in a parabolic partial

differential equation.
Consider the reaction diffusion equation

$$\frac{\partial u}{\partial t} = a(t)\frac{\partial^2 u}{\partial x^2} + \lambda(t)u + \phi(x,t), \quad x \in (0,1), \; t \in (0,T], \tag{3.5.1}$$

with initial condition

$$u(x,0) = f(x), \quad x \in [0,1], \tag{3.5.2}$$

and boundary conditions

$$u(0,t) = g_0(t), \quad t \in [0,T], \tag{3.5.3}$$

$$u(1,t) = g_1(t), \quad t \in [0,T], \tag{3.5.4}$$

subject to an extra measurement

$$u(x^*,t) = E(t), \quad t \in [0,T], \quad x^* \in (0,1), \tag{3.5.5}$$

where $\lambda(t)$ is a known function of t, $\phi(x,t)$ is a known function of x and t, f, g_0, g_1 and E are known functions, while $u(x,t)$ and $a(t)$ are unknown functions to be determined.

First, equation (3.5.1) is transformed into a non-classical problem.
Differentiating $E(t)$ with respect to t and using (3.5.1)-(3.5.5), we obtain

$$\begin{aligned} E^{'}(t) = \frac{\partial u(x^*,t)}{\partial t} &= a(t)\frac{\partial^2 u(x^*,t)}{\partial x^2} + \lambda(t)u(x^*,t) + \phi(x^*,t) \\ &= a(t)\frac{\partial^2 u(x^*,t)}{\partial x^2} + \lambda(t)E(t) + \phi(x^*,t). \end{aligned}$$

Assuming that $\frac{\partial^2 u(x^*,t)}{\partial x^2} \neq 0$ we have

$$a(t) = \frac{E^{'}(t) - \lambda(t)E(t) - \phi(x^*,t)}{\frac{\partial^2 u(x^*,t)}{\partial x^2}}, \quad t \in (0,T].$$

Therefore the inverse problem (3.5.1)-(3.5.5) is equivalent to the following problem:

$$\frac{\partial u}{\partial t} = \frac{E^{'}(t) - \lambda(t)E(t) - \phi(x^*,t)}{\frac{\partial^2 u(x^*,t)}{\partial x^2}}\frac{\partial^2 u}{\partial x^2} + \lambda(t)u + \phi(x,t), \quad (x,t) \in (0,1)\times[0,T], \tag{3.5.6}$$

$$u(x,0) = f(x), \quad x \in (0,1), \tag{3.5.7}$$

$$u(0,t) = g_0(t), \quad t \in (0,T], \tag{3.5.8}$$

$$u(1,t) = g_1(t), \quad t \in (0,T]. \tag{3.5.9}$$

Applying $\frac{\partial^2}{\partial x^2}$ to both sides of (3.5.6) and letting $v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$, we get

$$\frac{\partial v}{\partial t} = \frac{E^{'}(t) - \lambda(t)E(t) - \phi(x^*,t)}{v(x^*,t)}\frac{\partial^2 v}{\partial x^2} + \lambda(t)v + \frac{\partial^2 \phi(x,t)}{\partial x^2}, \quad (x,t) \in (0,1)\times[0,T], \tag{3.5.10}$$

$$v(x,0) = f^{''}(x), \quad x \in (0,1), \tag{3.5.11}$$

$$v(0,t) = \frac{(g_0^{'}(t) - \lambda(t)g_0(t) - \phi(0,t))v(x^*,t)}{E^{'}(t) - \lambda(t)E(t) - \phi(x^*,t)}, \quad t \in (0,T], \tag{3.5.12}$$

$$v(1,t) = \frac{(g_1^{'}(t) - \lambda(t)g_1(t) - \phi(1,t))v(x^*,t)}{E^{'}(t) - \lambda(t)E(t) - \phi(x^*,t)}, \quad t \in (0,T]. \tag{3.5.13}$$

Hence, the fourth order numerical scheme is applied to solve the problem (3.5.10)-(3.5.13).

The domain $[0,1] \times [0,T]$ is divided into an $M \times N$ mesh with the spatial step size $h = \frac{1}{M}$ in x direction and the time step size $\Delta t = \frac{T}{N}$, respectively. Grid points (x_i, t_j) are defined by

$$x_i = i * h, \quad i = 0,1,2,...,M, \tag{3.5.14}$$

$$t_n = n * \Delta t, \quad n = 0,1,2,...,N, \tag{3.5.15}$$

in which M and N are integers. The notations u_i^n and a^n are used to denote the finite difference approximations of $u(i*h, n*\Delta t)$ and $a(n*\Delta t)$, respectively.

First, by applying forward time centered space to equations (3.5.10)-(3.5.13). So, we have

$$\frac{\partial v}{\partial t} = \frac{v_{i,n+1} - v_{i,n}}{\Delta t} = \frac{v_i^{n+1} - v_i^n}{\Delta t},$$

and

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= \frac{v_{i+1,n} - 2v_{i,n} + v_{i-1,n}}{(\Delta x)^2} + O(h) \\ &= \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h^2} + O(h).\end{aligned}$$

Hence, we obtain

$$\frac{v_i^{n+1}-v_i^n}{\Delta t}=\frac{E^{'}(t_n)-\lambda(t_n)E(t_n)-\phi(x^*,t_n)}{h^2v(x^*,t_n)}(v_{i+1}^n-2v_i^n+v_{i-1}^n)+\lambda(t_n)v_i^n+\phi_{xx}(x_i,t_n). \tag{3.5.16}$$

Now, by applying backward-time centered space to problem (3.5.10)-(3.5.13) we obtain the same result as in (3.5.16) except that the spatial derivative on the right hand side are evaluated at time step $n+1$.

Simply from the average of the explicit and implicit finite difference schemes obtained above we get the Crank Nicolson algorithm

$$\frac{v_i^{n+1}-v_i^n}{\Delta t}=\frac{1}{2}[\frac{Q^{n+1}}{h^2v(x^*,t_{n+1})}\delta_x^2v_i^{n+1}+\frac{Q^n}{h^2v(x^*,t_n)}\delta_x^2v_i^n+\lambda(t_{n+1})v_i^{n+1}+ \\ \lambda(t_n)v_i^n+\phi_{xx}(x_i,t_{n+1})+\phi_{xx}(x_i,t_n)], \tag{3.5.17}$$

where $Q^n=E^{'}(t_n)-\lambda(t_n)E(t_n)-\phi(x^*,t_n)$ and $\delta_x^2v_i=v_{i+1}-2v_i+v_{i-1}$.

Using the fourth-order Pade approximation to approximate $\frac{\partial^2 v}{\partial x^2}$ in (3.5.10) we get

$$\frac{v_i^{n+1}-v_i^n}{\Delta t}=\frac{1}{2}[\frac{Q^{n+1}}{h^2v(x^*,t_{n+1})}\frac{\delta_x^2}{1+\frac{\delta_x^2}{12}}v_i^{n+1}+\frac{Q^n}{h^2v(x^*,t_n)}\frac{\delta_x^2}{1+\frac{\delta_x^2}{12}}v_i^n+\lambda(t_{n+1})v_i^{n+1}+ \\ \lambda(t_n)v_i^n+\phi_{xx}(x_i,t_{n+1})+\phi_{xx}(x_i,t_n)], \tag{3.5.18}$$

where this formula is second-order accurate in time but fourth-order accurate in space.

Applying $1+\frac{\delta_x^2}{12}$ to both sides of (3.5.18), we obtain the following scheme

$$[(1-\frac{\lambda(t_{n+1})\Delta t}{2})(1+\frac{\delta_x^2}{12})-\frac{\Delta tQ^{n+1}}{2h^2v(x^*,t_{n+1})}\delta_x^2]v_i^{n+1}=[(1+\frac{\lambda(t_n)\Delta t}{2})(1+\frac{\delta_x^2}{12})+\frac{\Delta tQ^n}{2h^2v(x^*,t_n)}\delta_x^2]v_i^n+ \\ \frac{\Delta t}{2}(1+\frac{\delta_x^2}{12})[\phi_{xx}(x_i,t_{n+1})+\phi_{xx}(x_i,t_n)]. \tag{3.5.19}$$

The term $v(x^*,t)$ is not explicitly defined in (3.5.19) therefore a fourth-order linear approximation to $v(x^*,t)$ is developed .
Hence, there are four different cases depending on the location of x^*.

case1: $x^* = x_i$ for some $1 \leq i \leq M-1$, $v(x^*, t_{n+1}) = v_i^{n+1}$ and $v(x^*, t_n) = v_i^n$.

case2: If $x^* \in (x_0, x_1)$, then $x^* = x_0 - \alpha h$ for some $\alpha \in (0,1)$ and $v(x^*, t)$ is approximated as

$$v(x^*, t) = c_1 v(x_0, t) + c_2 v(x_1, t) + c_3 v(x_2, t) + c_4 v(x_3, t), \qquad (3.5.20)$$

where $c_1,\ c_2,\ c_3$ and c_4 are coefficients to be determined.
By expanding $v(x_i, t)$ at x^* by Taylor series for $i = 0, 1, 2, 3$ and substituting these Taylor series in (3.5.16) and then matching the coefficients of $v(x^*, t)$, $v_x(x^*, t)$, $v_{xx}(x^*, t)$ and $v_{xxx}(x^*, t)$ on both sides of (3.5.16) a linear algebraic system is obtained and it's solution yields the coefficients from c_1 to c_4.

case3: If $x^* \in (x_{M-1}, x_M)$, the same method as case 2 can be used.

case4: If $x_l < x^* < x_{M-1}$, then there exists an integer l such that $x^* \in (x_l, x_{l+1})$ and both x_l and x_{l+1} are interior grid points, so $x^* = x_l + \alpha h$ for some $\alpha \in (0,1)$, then the linear approximation to $v(x^*, t)$ is given as

$$v(x^*, t) = c_1 v(x_{l-1}, t) + c_2 v(x_l, t) + c_3 v(x_{l+1}, t) + c_4 v(x_{l+2}, t), \qquad (3.5.21)$$

then by expanding $v(x_{l-1}, t)$, $v(x_l, t)$, $v(x_{l+1}, t)$ and $v(x_{l+2}, t)$ at $x = x^*$ by Taylor series and substituting these Taylor series in (3.5.17), the coefficients $c_1,\ c_2,\ c_3$ and c_4 are obtained.

Finally, it remains to solve $u(x, t)$ from $v(x, t)$ for a given t and this can be done by solving the following boundary value problem

$$u_{xx}(x, t) = v(x, t), \quad x \in (0, 1)$$

$$u(0, t) = g_0(t),$$
$$u(1, t) = g_1(t).$$

Simply, the above problem can be solved using the following finite difference scheme

$$\frac{\delta_x^2 u_i^n}{h^2} = v_i^n, \quad 1 \leq i \leq M-1,\ n \geq 1,$$
$$u_0^n = g_0(t_n), \quad n \geq 1,$$

$$u_M^n = g_1(t_n), \quad n \geq 1,$$

where v_i^n is the numerical solution of (3.5.19).

The above scheme is second-order accurate. It can be enhanced to obtain fourth-order accuracy by applying Pade approximation to u_{xx}, then we have

$$\frac{\delta_x^2}{h^2(1+\frac{\delta_x^2}{12})} u_i^n = v_i^n,$$

Applying $1 + \frac{\delta_x^2}{12}$ to both sides of the above equation, we obtain the following fourth-order scheme

$$u_{i-1}^n - 2u_i + u_{i+1}^n = h^2(v_{i-1}^n + 10v_i + v_{i+1}^n), \quad 1 \leq i \leq M-1, \ n \geq 1,$$

with boundary conditions $u_0^n = g_0(t_n)$ and $u_M^n = g_1(t_n)$.

Chapter 4

Inverse parabolic problems with discontinuous principal coefficient

4.1 Discontinuous principal coefficient

All the results related to uniqueness and stability of inverse parabolic problems that have been described in chapter two are applicable only to continuous coefficients of the principal part of an equation in divergent form.

Here in this section we consider the discontinuous coefficient $a = a^0 + b\chi(Q^*)$ where a^0, b are C^2-smooth functions and $\chi(Q^*)$ is the characteristic function of the unknown domain Q^*.
We will describe uniqueness results for time-independent and time-dependent Q^* with given one or many lateral boundary measurements.

In the next theorem Γ is any open subset of the hyperplane $x_n = 0$ and Ω is the half-space $\{x_n < 0\}$ in $\mathbb{R}^n$.
The following theorem formulate the uniqueness result for domains Q^* independent of time.

Theorem 4.1.1. [1] *Let $a^0 = 1$, and $Q^* = D \times (0, T)$, where D is a bounded Lipschitz x_n-convex domain in the half-space $\{x_n < 0\}$, $n \geq 2$, intersecting the strip $\Gamma \times \mathbb{R}_-$.*
Assume that k is known constant and $-1 < k < 0$. Let the Dirichlet lateral

data $g(x,t)$ be $g^0(x)\psi(t)$, where ψ corresponds via

$$u(x,t) = 2(2\pi t)^{\frac{-1}{2}} \int_0^\infty \exp(\frac{-\tau^2}{4t}) u^*(x,\tau) d\tau,$$

to a function $\psi^*(\tau)$ that is positive as well as it's first-order derivative on some interval $(0,\epsilon)$, $|\psi_0(\tau)| \leq Cexp(C\tau)$ and $g_0 = 1$ on Γ, $g_0 \in C_0^2(\partial\Omega)$.
Let u be a solution to the parabolic problem (2.0.1)-(2.0.3) with $a = a^0 + b\chi(Q^*)$ and with $a_0 = 1$, $b = 0$, $c = 0$, $u_0 = 0$.
Then

$$\frac{\partial u}{\partial v} \text{ on } \Gamma \times (0,T),$$

uniquely determines D.

Consider the second-order parabolic equation with discontinuous principal coefficient:

$$\frac{\partial u_j}{\partial t} - div(a_j \nabla u_j) = 0 \;\; in \;\; Q = \Omega \times (0,T), \tag{4.1.1}$$

with the initial and boundary conditions

$$u = 0 \qquad on \qquad \Omega \times \{0\}, \tag{4.1.2}$$

$$u_j = g \;\; on \;\; S = \partial\Omega \times (0,T), \tag{4.1.3}$$

when $\frac{\partial u_j}{\partial \nu}$ is given for all (regular) g.
Here Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with the boundary $\partial\Omega \in C^2$.

We are interested in finding an open set Q_j and a function b_j for this problem where $a_j = a_0 + \chi(Q_j) b_j > \epsilon > 0$, $b_j \neq 0$ on ∂Q_j.

It's well known that for any $g \in C^{1,2}(\bar{S})$, $g = g_t = g_{tt} = 0$ *on* $\partial\Omega \times \{0\}$, there is a unique (generalized) solution u_j of this problem and $u_j \in C^\lambda(\bar{Q})$ for some $\lambda \in (0,1)$, $\nabla_x u_j \in L_2(Q)$, and $\in C(\bar{Q} \backslash \bar{Q}_j)$.

We will review the proof of uniqueness of discontinuous coefficient $a_j = a_0 + b_j \chi(Q_j^*)$, where $\chi(Q_j^*)$ is the characteristic function of an open set $Q_j^* \subset Q$ with the Lipschitz lateral boundary $\partial_x Q_j^*$ changing with time and $a_0 = a_0(x)$ and $b_j = b_j(x)$ are respectively, given and unknown $C^2(\bar{\Omega})-$functions.

In the next theorem Γ_0 is $\partial\Omega \bigcup B_0$ for some ball B_0 centered at a point of $\partial\Omega$.

Theorem 4.1.2. [7] *Suppose Q_1 and Q_2 are open x-Lipschitz sets, $Q_j \subset \Omega \times (-T, 2T)$, and*

$$\text{the sets } \ (Q \backslash \bar{Q}_j) \bigcap \{t = \tau\} \ \text{ are connected when } 0 < \tau < T. \tag{4.1.4}$$

If solutions u_j to the initial-boundary value problems (4.1.1), (4.1.2), and (4.1.3) satisfy the equality

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \quad on \quad \Gamma_0 \times (0, T) \quad (\nu \text{ is a normal}), \tag{4.1.5}$$

for all $g \in C^2(\partial\Omega \times [0, T])$ with $suppg \subset \Gamma_0 \times (0, T)$, then

$$a_1 = a_2 \quad on \quad Q. \tag{4.1.6}$$

This result guarantees uniqueness of reconstruction of Q_j from all possible lateral measurements for an arbitrary $T > 0$.

Before proving theorem (4.1.2) we need to obtain some auxiliary relations which well be used in it's proof.

Denote by Q_{3t} the connected components of the open set $\Omega \backslash (\bar{Q}_{1t} \bigcup \bar{Q}_{2t})$ whose boundary contains Γ_0. Here $Q_{j\theta}$ is $Q_j \bigcap \{t = \theta\}$, $j = 1, 2$. Let $Q_3 = \bigcup Q_{3t}$ over $0 < t < T$ and let $Q_4 = Q \backslash \bar{Q}_3$.

lemma 4.1.1. [7] *Under the conditions of theorem (4.1.2) we have the following orthogonality relations:*

$$\int_{Q_1} b_1 \nabla v_1 . \nabla u_2^* \; dxdt = \int_{Q_2} b_2 \nabla v_1 . \nabla u_2^* \; dxdt, \tag{4.1.7}$$

for all solutions v_1 to equation (4.1.1)($j = 1$) near $\bar{Q}_4$ that are 0 when $t < 0$ and solutions u_2^* to the adjoint equation $\frac{\partial u_2^*}{\partial t} + div(a_2 \nabla u_2^*) = 0$ near $\bar{Q}_4$ that are 0 when $t > T$.

Proof. From well-known results about regularity of solutions to the Dirichlet problem (4.1.1), (4.1.2) it follows that $u_j \in C^{2,1}(Q_3)$ and in $H^{2,1}(Q_5)$, where $Q_5 = V \times (0, T)$ and V is a vicinity of $\partial\Omega$ in Ω.
Due to conditions (4.1.3) and (4.1.1), both u_1 and u_2 have the same Cauchy

data on $\Gamma_0 \times (0,T)$ and satisfy the same parabolic equation in Q_3; thus from uniqueness of continuation for second-order parabolic equations, we conclude that u_1 and u_2 coincide on Q_3.

Letting $u = u_2 - u_1$ and subtracting the equations (4.1.1) with $j = 1$ from those with $j = 2$, we get

$$div((a_0 + b_2\chi(Q_2))\nabla u) - \frac{\partial u}{\partial t} = div((b_1\chi(Q_1) - b_2\chi(Q_2))\nabla u_1) \ \ in\ Q. \quad (4.1.8)$$

Now using the definition of a weak solution to the parabolic equation (4.1.8) we obtain

$$\int_Q ((a_0+b_2\chi(Q_2))\nabla u.\nabla\psi + \frac{\partial u}{\partial t}\psi)\, dxdt = \int_Q (b_1\chi(Q_1) - b_2\chi(Q_2))\nabla u_1.\nabla\psi\, dxdt, \quad (4.1.9)$$

for all test functions ψ from $H_0^{1,1}(Q)$ that are equal zero on $\Omega \times \{T\}$.

Since u and $\chi(Q_j)$ are zero outside $\bar{Q}_4$, this relation remains valid for any function ψ from $H^{1,1}(Q_5)$, where Q_5 is an arbitrary vicinity of $\bar{Q}_4$.

Now integrating the left side of (4.1.9) by parts with respect to t and using that $\psi(,T) = 0$ we get

$$\int_Q (a_0 + b_2\chi(Q_2))\nabla u.\nabla\psi - \int_Q \frac{\partial\psi}{\partial t}u. \quad (4.1.10)$$

Using the definition of a weak solution to the adjoint equation in lemma (4.1.1) with the test function u (which is zero outside $Q_4 \bigcap \{t < T\}$)we obtain

$$\int_Q ((a_0 + b_2\chi(Q_2))\nabla u_2^*.\nabla u + \frac{\partial u_2^*}{\partial t}u) = 0. \quad (4.1.11)$$

If $\psi = u_2^*$ is a solution to this adjoint equation, then we conclude from (4.1.10) and (4.1.11) that the left side in (4.1.9) is zero.
Thus, we have

$$\int_{Q_1} b_1\nabla u_1.\nabla u_2^*\, dxdt = \int_{Q_2} b_2\nabla u_1.\nabla u_2^*\, dxdt. \quad (4.1.12)$$

The final relation can be obtained by using the approximation of v_1 by u_1.

Now by using the Runge property, equality (4.1.7) can be extended onto all v_1 solving equation (4.1.1) with $j = 1$ near $\bar{Q}_4$ and satisfying the initial condition (4.1.2). Denote the space of such v_1 by X. Letting X_1 be the space of solutions to the Dirichlet problem (4.1.1)-(4.1.3) with $j = 1$ for various g supported in $\Gamma_0 \times (0, T)$.

Let $u_1 \in X_1$ then it's sufficient to prove that solutions in X_1 approximate in $L_2(Q_4)$ any solution from X.
Indeed, let $v_1 \in X$. Then we can approximate it similarly by solutions from X in $L_2(Q_7)$, where Q_7 is a Lipschitz domain containing Q_4 with $dis(\partial_x Q_7, Q_4) > 0$. From the well-known interior Schauder-type estimates for parabolic equations, it follows that these solutions from X_1 well approximate v_1 in $H^{1,0}(Q_4)$.

To prove L_2 approximation in view of the Hahn-Banach theorem, it's sufficient to show that if f from the dual space $L_2(Q_4)$ is orthogonal to X_1, then f is orthogonal to X.

Let Ω_0 be a Lipschitz bounded domain containing Ω that is not equal to Ω but such that $\partial\Omega \backslash \Gamma_0 \subset \partial\Omega_0$. Let $K(x, t; y, s)$ be the Green's function of the Dirichlet problem for the operator $\frac{\partial}{\partial t} + div(a_1 \nabla)$ in $Q_0 \times (0, T)$. Let f be orthogonal to X_1.

The Green potential

$$U(x, t; f) = \int_{Q_4} f K(x, t;), \tag{4.1.13}$$

is equal to zero on $Q_0 \backslash \bar{Q}_4$ because the function $u_1 = K(x, t;)$ belongs to X_1 if $(x, t) \in Q_0 \backslash \bar{Q}_4$. The last function $U(x)$ is the volume potential with density f supported in $\bar{Q}_4$, so it is a solution to the equation $-div(a_0 \nabla u) = \frac{\partial u}{\partial t}$ on $Q_0 \backslash \bar{Q}_4$. Since it is zero on $Q_0 \backslash \bar{Q}_4$ and a_0 belongs to $C^1(\bar{Q}_0)$, we can use uniqueness of the continuation and conclude that $U(; f) = 0$ on $Q_0 \backslash \bar{Q}_4$.

Now let $v \in X$; then v is a solution to the homogeneous equation near $Q_5 \bigcup \partial_x Q_4$, where Q_5 is an open set with C^∞ lateral boundary and $dis(\partial_x Q_5, \partial_x Q_4) > 0$.

Using the representation of v by a single layer potential, we obtain

$$v(y,s)=\int_{\partial_x Q_5} gK(;y,s)d\Gamma,$$

for some $g\in C(\partial_x Q_5)$.
By using this representation, (4.1.13), and Fubini's theorem, we obtain

$$\int_{Q_4} fv=\int_{\partial_x Q_5} gU(;f)=0$$

because $U(;f)=0$ on $\partial_x Q_5$.

So, if $f=0$ on X_1, then $f=0$ on X, and so X_1 is dense in X.
Accordingly, relation(4.1.7) is valid for any v_1 satisfying the conditions of lemma (4.1.1). □

Returning to the proof of theorem (4.1.2).

First we will show that $Q_1=Q_2$. Assume the opposite: Q_1 is not contained in Q_2. By using condition (4.1.4) of theorem (4.1.2) we can find a point $(x_0,t_0)\in\partial Q_1\backslash\bar{Q}_2$ such that $(x_0,t_0)\in\partial_x Q_3$. We may assume that this point is the origin. Choose the ball B so that it's closure is contained in Ω and a cylinder $Z=B\times(0,\tau)$ such that $\bar{Z}$ doesn't intersect $\bar{Q}_2$, and $(\partial_x Q_1)\bigcap\bar{Z}$ is a Lipschitz surface. By the extension theorem there is a function $a_3\in C^2(\bar{Q}_1\bigcup\bar{Z})$ that agrees with a_1 on Q_1. Extend a_3 onto $Q\backslash\bar{Q}_1\bigcup\bar{Z}$ as a_0.
To complete the proof we need the following modification of the orthogonality relations (4.1.7).

lemma 4.1.2. [7] *Under the conditions of lemma (4.1.1),*

$$\int_{Q_1} b_1\nabla u_3.\nabla u_2^*=\int_{Q_2} b_2\nabla u_3.\nabla u_2^*,$$

for any solution u_3 to the parabolic equation

$$\frac{\partial u_3}{\partial t}-div(a_3\nabla u_3)=0\quad near\quad \bar{Q}_4,$$

$$u_3=0\quad when\quad t<0,$$

and for any solution u_2^* to the parabolic equation from lemma (4.1.1).

Proof. Consider u_3 and let Q_8 be an open set with $C^\infty-$boundary $\partial_x Q_8$ and $Q_4 \subset Q_8$ with $dis(\partial_x Q_8, Q_4) > 0$ such that u_3 is a solution to the equation $\frac{\partial u_3}{\partial t} - div(a_3 \nabla u_3) = 0$ near $\bar{Q}_8$.

Introduce a sequence of open sets Q_{4k} such that
i)$Q_{4k} \backslash Z = Q_4 \backslash Z$,
ii)the (Hausdorff) distance from ∂Q_{4k} to $\partial_x Q_4$ is less than $\frac{1}{k}$,
iii)$\partial_x Q_{4k} \bigcap Z$ doesn't intersect $\bar{Q}_4$.

Define a coefficient a_{3k} as a_3 on $Q_8 \backslash (Q_{4k} \backslash Q_4)$ and as a_0 on $Q_{4k} \backslash Q_4$.

Since $\partial Q_4 \bigcap Z$ is a lipschitz surface, we have

$$\lim_{k \to +\infty} meas_n \{a_{3k} \neq a_3\} = 0, \tag{4.1.14}$$

where $meas_n$ stands for the n-dimensional Lebesgue measure.
Let u_{3k} be solutions to the initial boundary value problems:

$$\frac{\partial u_{3k}}{\partial t} - div(a_{3k} \nabla u_{3k}) = 0 \;\; in \; Q_8,$$

$$u_{3k} = u_3 \;\; on \; \partial_x Q_8,$$

$$u_{3k} = 0 \;\; on \; Q_8 \bigcap \{t = 0\}.$$

Since $a_{3k} = a_0 + \chi(Q_1) b_1$ near $\bar{Q}_1$, relation (4.1.7) is valid for any $u_1 = u_{3k}$. Subtracting the equations (4.1.1) with $j = 3$ from those with $j = 3k$ we get

$$\frac{\partial}{\partial t}(u_{3k} - u_3) - div[a_{3k} \nabla u_{3k} - a_3 \nabla u_3] = 0.$$

Adding and subtracting the term $a_{3k} \nabla u_3$ we have

$$\frac{\partial}{\partial t}(u_{3k} - u_3) - div[a_{3k} \nabla u_{3k} - a_{3k} \nabla u_3 + a_{3k} \nabla u_3 - a_3 \nabla u_3] = 0.$$

Letting $u_k = u_{3k} - u_3$ we get

$$\frac{\partial u_k}{\partial t} - div[a_{3k} \nabla u_k + (a_{3k} - a_3) \nabla u_3] = 0.$$

Then the difference $u_k = u_{3k} - u_3$ satisfies the equation

$$\frac{\partial u_k}{\partial t} - div(a_{3k} \nabla u_k) = div((a_{3k} - a_3) \nabla u_3) \;\; in \; Q_8.$$

Now both u_{3k} and u_3 coincide on the lateral boundary of Q_8 and when $t = 0$, hence $u_k = 0$ on $\partial Q_8 \bigcap \{t < T\}$.
So, we have the Dirichlet problems

$$\frac{\partial u_k}{\partial t} - div(a_{3k}\nabla u_k) = div(a_{3k} - a_3)\nabla u_3 \;\; in \; Q_8,$$

$$u_k = 0 \;\; on \; \partial Q_8 \bigcap \{t < T\}.$$

From the definition of a weak solution to this initial boundary value problem with the test function u_k we have

$$-\int_{Q_8} a_{3k}\nabla u_k.\nabla u_k - \int_{Q_8} u_{kt}u_k = -\int_{Q_8} (a_3 - a_{3k})\nabla u_3 \nabla u_k.$$

Now, integrating by parts with respect to t, we obtain

$$\int_{Q_8} u_{kt}u_k = \int_{Q_8} u_k(,T)u_k(,T) - u_k(,0)u_k(,0) - \int_{Q_8} u_{kt}u_k,$$

$$2\int_{Q_8} u_{kt}u_k = \int_{Q_8} u_k^2(,T).$$

So,

$$\int_{Q_8} u_{kt}u_k = \frac{1}{2}\int_{Q_8 \bigcap \{t=T\}} u_k^2.$$

Finally, we obtain

$$\int_{Q_8} a_{3k}\nabla u_k.\nabla u_k + \frac{1}{2}\int_{Q_8 \bigcap \{t=T\}} u_k^2 = \int_{Q_8} (a_3 - a_{3k})\nabla u_3.\nabla u_k.$$

According to the assumption about a_j we have $a_{3k} > \epsilon$ for some $\epsilon > 0$ therefore the second integral on the left can be dropped.
Now, using the relation $x.y \leq \frac{1}{2\epsilon}|x|^2 + \frac{\epsilon}{2}|y|^2$ and holder inequality the right side of the above integral is bounded as follow

$$\int_{Q_8} \epsilon |\nabla u_k|^2 \leq C(\epsilon)\int_{Q_8} |a_3 - a_{3k}|^2 |\nabla u_3|^2 + \frac{\epsilon}{2}\int_{Q_8} |\nabla u_k|^2 . \qquad (4.1.15)$$

Since ∇u_3 belongs to $L_2(Q_8)$, we conclude from (4.1.14) that the first integral in the right side in (4.1.15) tends to 0. Therefore, ∇u_k converges to 0 in

$L_2(Q_8)$.

Putting $v_1 = u_{3k} = u_3 + u_k$ into (4.1.7) we have

$$\int_{Q_1} b_1 \nabla(u_3 + u_k) \nabla u_2^* \; dxdt = \int_{Q_2} b_2 \nabla(u_3 + u_k) \nabla u_2^* \; dxdt.$$

Letting $k \to \infty$, we get

$$\int_{Q_1} b_1 \nabla u_3 \nabla u_2^* \; dxdt = \int_{Q_2} b_2 \nabla u_3 \nabla u_2^* \; dxdt.$$

$\square$

To obtain a contradiction with the assumption $Q_1 \neq Q_2$, the solutions u_3 and u_2^* with singularities outside Q_4 can be used. To simplify obtaining bounds on the integrals of such solution it is convenient to use new variables. We can assume that the direction e_n of the x_n-axis coincides with the interior unit normal to $\partial_x Q_1 \bigcap \{t = 0\}$.

This surface near the origin is the graph of a Lipschitz function $x_n = q_1(x_1, ..., x_{n-1}, t)$ which can be assumed to be defined and Lipschitz on the whole $\mathbb{R}^n$.

The substitution

$$x_k = x_k^*, \quad k = 1, ..., n-1 \quad x_n = x_n^* + q_1(x_1^*, ..., x_{n-1}^*, t), \quad t = t^*,$$

transforms the equations (4.1.1) into similar ones with additional first-order differentiation with respect to x_n^* multiplied by a Lipschitz function of t. The domains Q_j are transformed into domains with similar properties and with the additional property that the points $(0, t)$, $0 < t < T$, belong to $\partial_x Q_1$.

Since the (hyper) plane $\{x_n^* = 0\}$ is tangent to this surface at the origin, we can find a cone $\mathcal{C} = \{\left|\frac{x^*}{|x^*|} - e_n\right| < \theta, \; |x^*| < \epsilon\}$ such that the cylinder $\mathcal{C} \times (0, T)$ is inside Q_1. Henceforth, we drop the sign $*$.

Let K^+ be the fundamental solution of the Cauchy problem for the forward parabolic equation

$$\frac{\partial u_3}{\partial t} - div(a_3 \nabla u_3) = 0,$$

and let K^- be the fundamental solution of the backward Cauchy problem for the backward parabolic equation

$$\frac{\partial u_2}{\partial t} + div(a_2 \nabla u_2) = 0,$$

with coefficients a_3 and a_2 with poles at the points $(y, 0)$, (y, τ), where y is outside $\bar{Q}_4$ and is close to the origin.

The singular(fundamental) solutions to be used have the following structure

$$K^+ = K_1^+ + K_0^+, \qquad K^- = K_1^- + K_0^-, \tag{4.1.16}$$

where K_1^+ and K_1^- are the principal parts of K^+ K^- and K_0^+ and K_0^- are the remainders with weaker singularities.

The principal parts are

$$K_1^+(x,t;y,\tau) = \frac{C}{(a_3(y)(t-\tau))^{\frac{n}{2}}} \exp(\frac{-|x-y|^2}{4a_3(y)(t-\tau)}),$$

$$K_1^-(x,t;y,\tau) = \frac{C}{(a_0(y)(\tau-t))^{\frac{n}{2}}} \exp(\frac{-|x-y|^2}{4a_0(y)(\tau-t)}). \tag{4.1.17}$$

From the known bounds of fundamental solutions of parabolic equations, we have

$$\left|\nabla_x K_0^+(x,t;y,\tau)\right| \le C(t-\tau)^{\frac{-n}{2}} \exp(\frac{-|x-y|^2}{(C(t-\tau))}),$$

$$\left|\nabla_x K_0^-(x,t;y,\tau)\right| \le C(\tau-t)^{\frac{-n}{2}} \exp(\frac{-|x-y|^2}{(C(\tau-t))}). \tag{4.1.18}$$

When $(y,0)$ and (y,τ) are outside $\bar{Q}_1$, the functions $K^+(;y,0)$ and $K^-(;y,\tau)$ are $(x,t)-$solutions to the homogeneous parabolic equations with bounded measurable coefficients satisfying zero initial and final conditions.

Letting $u_3 = K^+(;y,0)$ and $u_2^* = K^-(;y,\tau)$ in lemma (4.1.2) we get

$$\int_{Q_1} b_1 \nabla_x K^+(;y,0).\nabla_x K^-(;y,\tau) = \int_{Q_2} b_2 \nabla_x K^+(;y,0).\nabla_x K^-(;y,\tau).$$

Breaking the integration domain Q_1 into $Q_1 \bigcap Z$ and it's complement and using that $Q_1 \backslash Z = Q_1 \bigcap \bar{Z}$ we obtain

$$\int_{Q_1 \bigcap Z} b_1 \nabla_x K^+(;y,0).\nabla_x K^-(;y,\tau) + \int_{Q_1 \backslash Z} b_1 \nabla_x K^+(;y,0).\nabla_x K^-(;y,\tau) =$$

$$\int_{Q_2} b_2 \nabla_x K^+(;y,0).\nabla_x K^-(;y,\tau).$$

Then,

$$\int_{Q_1 \bigcap Z} b_1 \nabla_x K^+(;y,0).\nabla_x K^-(;y,\tau) = -\int_{Q_1 \backslash Z} b_1 \nabla_x K^+(;y,0).\nabla_x K^-(;y,\tau)$$

$$+\int_{Q_2} b_2 \nabla_x K^+(;y,0).\nabla_x K^-(;y,\tau). \tag{4.1.19}$$

From the estimates in (4.1.18) and similar estimates for $\nabla_x K_1^+$ and $\nabla_x K_1^-$, we conclude that the integrands are bounded by an integrable function uniformly with respect to y outside Q_1.
By the Lebesgue dominated-convergence theorem, we may let $y \longrightarrow 0$ and replace y in (4.1.19) by 0 (the integrals became functions of τ only). Using representation (4.1.16), we obtain from (4.1.19) that

$$|I_1| \leq |I_2| + |I_3|, \tag{4.1.20}$$

where

$$I_1 = \int_{Q_1 \bigcap Z} b_1 \nabla_x K_1^+(;0,0).\nabla_x K_1^-(;0,\tau),$$

$$I_2 = -\int_{Q_1 \backslash Z} b_1 \nabla_x K^+(;0,0).\nabla_x K^-(;0,\tau) + \int_{Q_2} b_2 \nabla_x K^+(;0,0).\nabla_x K^-(;0,\tau),$$

and

$$I_3 = \int_{Q_1 \bigcap Z} b_1 (\nabla_x K_1^+(;0,0).\nabla_x K_0^-(;0,\tau) + \nabla_x K_0^+(;0,0).\nabla_x K_1^-(;0,\tau) + \nabla_x K_0^+(;0,0).\nabla K_0^-(,0;\tau)).$$

lemma 4.1.3. [7] *We have*

$$C^{-1}\tau^{-n}\int_0^{\epsilon}\rho^{n-1}\exp(\frac{-4\rho^2}{m\tau})\leq |I_1|\,,$$

$$|I_2|\leq C\tau^{\frac{-n}{2}+1}\epsilon^{-2}\exp(\frac{-\epsilon^2}{M\tau}),$$

$$|I_3|\leq C\epsilon\tau^{\frac{-n}{2}},$$

where the constant M depends only on the upper bounds of a_3, a_0 over Q, the constant C doesn't depend on τ, and the constant m depends only on the lower bounds of $a_3,\ a_0$ over Q.

Proof. Consider $I_1=\int_{Q_1\bigcap Z}b_1\nabla_x K_1^+(;0,0).\nabla_x K_1^-(;0,\tau)dxdt$.
Using the fact that $b_1(0)\neq 0$ and choosing ϵ in the definition of $\mathcal{C}$ to be sufficiently small, we obtain

$$|I_1|\geq\frac{1}{C}\int_{\mathcal{C}\times(0,\tau)}\nabla_x K_1^+(x,t;0,0)\nabla_x K_1^-(x,t;0,\tau)dxdt,$$

$$=\frac{1}{C}\int_0^{\tau}\int_{\mathcal{C}}\frac{x}{t^{\frac{n}{2}+1}}\exp(\frac{-|x|^2}{a_3(x)t}).\frac{x}{(\tau-t)^{\frac{n}{2}+1}}\exp(\frac{-|x|^2}{a_0(x)(\tau-t)})dxdt,$$

$$\geq\frac{1}{C}\int_{\mathcal{C}}\int_0^{\frac{\tau}{2}}\frac{|x|^2}{((\tau-t)t)^{\frac{n}{2}+1}}\exp(\frac{-|x|^2\tau}{mt(\tau-t)})dtdx.$$

Using the inequality

$$\frac{1}{t\tau}\leq\frac{1}{t(\tau-t)}\leq\frac{2}{t\tau}\quad when\quad 0<t<\frac{\tau}{2},$$

we bound from below the integral shown above by

$$\frac{1}{C}\int_{\mathcal{C}}\int_0^{\frac{\tau}{2}}\frac{|x|^2}{(\tau t)^{\frac{n}{2}+1}}\exp(\frac{-2|x|^2}{mt})dtdx,$$

$$=\frac{1}{C\tau^{\frac{n}{2}+1}}\int_{\mathcal{C}}\int_0^{\frac{\tau}{2}}\frac{|x|^2}{t^{\frac{n}{2}+1}}\exp(\frac{-2|x|^2}{mt})dtdx.$$

Letting $w = \frac{2|x|^2}{mt}$, then $t = \frac{2|x|^2}{mw}$ and hence $dt = \frac{-2|x|^2}{mw^2}dw$.
Now, when $t \to 0$ then $w \to \infty$ and when $t = \frac{\tau}{2}$ then $w = \frac{4|x|^2}{m\tau}$.
Substituting in the last integral, we get

$$\frac{1}{C\tau^{\frac{n}{2}+1}} \int_{\mathcal{C}} \int_{\frac{4|x|^2}{m\tau}}^{\infty} \frac{|x|^2}{(\frac{2|x|^2}{mw})^{\frac{n}{2}+1}} . \frac{2\,|x|^2}{mw^2} \exp(-w) dw dx,$$

$$= \frac{1}{C\tau^{\frac{n}{2}+1}} \int_{\mathcal{C}} \int_{\frac{4|x|^2}{m\tau}}^{\infty} |x|^{-n+2}\, w^{\frac{n}{2}-1} \exp(-w) dw dx.$$

The function $w^{\frac{n}{2}-1}$ is increasing, so replacing it by it's minimal value at $w = 4\frac{|x|^2}{m\tau}$, we bound the integral shown above from below by

$$\geq \frac{1}{C\tau^{\frac{n}{2}+1}} \int_{\mathcal{C}} \int_{\frac{4|x|^2}{m\tau}}^{\infty} |x|^{-n+2} \left(\frac{4\,|x|^2}{m\tau}\right)^{\frac{n}{2}-1} \exp(-w) dw dx,$$

$$= \frac{1}{C\tau^{\frac{n}{2}+1}} \int_{\mathcal{C}} \int_{\frac{4|x|^2}{m\tau}}^{\infty} \frac{1}{\tau^{\frac{n}{2}-1}} \exp(-w) dw dx,$$

$$= \frac{1}{C\tau^{n}} \int_{\mathcal{C}} \int_{\frac{4|x|^2}{m\tau}}^{\infty} \exp(-w) dw dx,$$

$$= \frac{1}{C\tau^{n}} \int_{\mathcal{C}} \exp\left(-\frac{4\,|x|^2}{m\tau}\right) dx.$$

Using polar coordinates we get

$$|I_1| \geq C^{-1}\tau^{-n} \int_0^{\epsilon} \rho^{n-1} \exp\left(\frac{-4\rho^2}{m\tau}\right),$$

I_2 consists of two integrals. Thus we have

$$I_2 = I_{21} + I_{22},$$

$$I_{21} = \int_{Q_1 \setminus Z} b_1 \nabla K^+ \nabla K^-,$$

and it's bounded above by

$$C \int_{\epsilon<|x|<R} \int_0^{\tau} \left|\nabla_x K^+(;0,0).\nabla_x K^-(;0,\tau)\right|,$$

$$\leq C\int_{\epsilon<|x|<R}\int_0^{\frac{\tau}{2}}\frac{1}{((\tau-t)t)^{\frac{n}{2}+1}}\exp(\frac{-|x|^2\tau}{Mt(\tau-t)})dtdx.$$

Using that

$$\frac{|x|}{(t-\tau)^{\frac{n}{2}+1}}\exp(\frac{-|x|^2}{4(t-\tau)})\leq\frac{C}{(t-\tau)^{\frac{n}{2}+\frac{1}{2}}}\exp(\frac{-|x|^2}{8(t-\tau)}),$$

Then, we have

$$C\int_{\epsilon<|x|<R}\int_0^{\frac{\tau}{2}}\frac{1}{((\tau-t)t)^{\frac{n}{2}+\frac{1}{2}}}\exp(\frac{-|x|^2\tau}{M't(\tau-t)}),\quad M^{'}=2M.$$

Using the inequality

$$\frac{1}{t\tau}<\frac{1}{(t-\tau)t}<\frac{2}{t\tau}\quad when\quad 0<t<\frac{\tau}{2}.$$

The last integral is bounded by

$$\frac{C}{\tau^{\frac{n}{2}+\frac{1}{2}}}\int_{\epsilon<|x|<R}\int_0^{\frac{\tau}{2}}\frac{1}{t^{\frac{n}{2}+\frac{1}{2}}}\exp(\frac{-|x|^2}{Mt})dtdx.$$

Letting $w=\frac{|x|^2}{Mt}$, then $t=\frac{|x|^2}{Mw}$ and $dt=\frac{-|x|^2}{Mw^2}$.
After we use this substitution we obtain

$$\leq\frac{C}{\tau^{\frac{n}{2}+\frac{1}{2}}}\int_{\epsilon<|x|<R}\frac{1}{(|x|)^{n-1}}\int_{\frac{2|x|^2}{M\tau}}^{\infty}w^{\frac{n}{2}-1}w^{-\frac{1}{2}}\exp(-w)dw.$$

The function $w^{-\frac{1}{2}}$ is decreasing. Replacing it by it's value at $\frac{2|x|^2}{M\tau}$ and using the inequality $w^{\frac{n}{2}-1}\exp(-w)\leq C\exp(-\frac{w}{2})$ we obtain

$$\leq\frac{C}{\tau^{\frac{n}{2}+\frac{1}{2}}}\int_{\epsilon<|x|<R}\frac{1}{(|x|)^{n-1}}\int_{\frac{2|x|^2}{M\tau}}^{\infty}(\frac{2|x|^2}{M\tau})^{-\frac{1}{2}}\exp(-\frac{w}{2})dwdx,$$

$$\leq\frac{C}{\tau^{\frac{n}{2}}}\int_{\epsilon<|x|<R}\frac{1}{(|x|)^{n}}\int_{\frac{2|x|^2}{M\tau}}^{\infty}\exp(-\frac{w}{2})dwdx,$$

$$=\frac{C}{\tau^{\frac{n}{2}}}\int_{\epsilon<|x|<R}\frac{1}{(|x|)^{n}}\exp(\frac{-|x|^2}{M\tau})dx.$$

Using the polar coordinates in $\mathbb{R}^n$, we get

$$\leq \frac{C}{\tau^{\frac{n}{2}}} \int_{\epsilon}^{\infty} \rho^{-n} \rho^{n-1} \exp(\frac{-\rho^2}{M\tau}) d\rho,$$

$$= \frac{C}{\tau^{\frac{n}{2}}} \int_{\epsilon}^{\infty} \rho^{-1} \exp(\frac{-\rho^2}{M\tau}) d\rho,$$

$$\frac{C}{\tau^{\frac{n}{2}}} \int_{\epsilon}^{\infty} \rho^{-2} \rho \exp(\frac{-\rho^2}{M\tau}) d\rho,$$

$$\leq \frac{C}{\tau^{\frac{n}{2}}} \frac{1}{\epsilon^2} \int_{\epsilon}^{\infty} \rho \exp(\frac{-\rho^2}{M\tau}) d\rho,$$

$$= \frac{C}{\tau^{\frac{n}{2}}} \frac{1}{\epsilon^2} \tau \exp(\frac{-\epsilon^2}{M\tau}),$$

$$= \frac{C}{\tau^{\frac{n}{2}-1}} \frac{1}{\epsilon^2} \exp(\frac{-\epsilon^2}{M\tau}).$$

So,

$$|I_2| \leq \frac{C}{\tau^{\frac{n}{2}-1}} \frac{1}{\epsilon^2} \exp(\frac{-\epsilon^2}{M\tau}).$$

The other term can be bounded in a similar way.
It remains to prove the last inequality, hence we start by

$$|I_{31}| \leq C \int \nabla_x K_1^+(;0,0) \nabla K_0^-(;0,\tau),$$

$$\leq C \int_{|x|<\epsilon} \int_0^{\frac{\tau}{2}} \frac{|x|}{t^{\frac{n}{2}+1}} \exp(\frac{-|x|^2}{a_3(x)t}). \frac{C}{(\tau-t)^{\frac{n}{2}}} \exp(\frac{-|x|^2}{C(\tau-t)}) dt dx.$$

Replacing $|x|$ by some power of $(t-\tau)$, we get

$$\leq C \int_{|x|<\epsilon} \int_0^{\frac{\tau}{2}} \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} \exp(\frac{-|x|^2}{a_3(x)t}). \frac{C}{(\tau-t)^{\frac{n}{2}}} \exp(\frac{-|x|^2}{C(\tau-t)}) dt dx,$$

$$\leq \int_{|x|<\epsilon} \int_0^{\frac{\tau}{2}} ((\tau-t)t)^{-\frac{n}{2}} t^{-\frac{1}{2}} \exp(\frac{-|x|^2 \tau}{Mt(\tau-t)}) dt dx,$$

$$\leq C \int_{|x|<\epsilon} \int_0^{\frac{\tau}{2}} \frac{t}{(\tau t)^{\frac{n}{2}}} t^{-\frac{1}{2}} \exp(\frac{-|x|^2}{Mt}) dt dx,$$

$$\leq \frac{C}{\tau^{\frac{n}{2}}} \int_{|x|<\epsilon} \int_0^{\frac{\tau}{2}} t^{-\frac{n}{2}-\frac{1}{2}} \exp(\frac{-|x|^2}{Mt}) dt dx.$$

Using the substitution $w = \frac{|x|^2}{Mt}$, the last integral is bounded above by

$$\frac{C}{\tau^{\frac{n}{2}}} \int_{|x|<\epsilon} \int_{\frac{2|x|^2}{M\tau}}^{\infty} \frac{|x|^{-n-1}}{w^{-\frac{n}{2}-\frac{1}{2}}} . \frac{|x|^2}{w^2} \exp(-w) dw dx,$$

$$= \frac{C}{\tau^{\frac{n}{2}}} \int_{|x|<\epsilon} \int_{\frac{2|x|^2}{M\tau}}^{\infty} |x|^{-n+1} w^{\frac{n}{2}-\frac{3}{2}} \exp(-w) dw dx,$$

$$= \frac{C}{\tau^{\frac{n}{2}}} \int_{|x|<\epsilon} |x|^{-n+1} \int_{\frac{2|x|^2}{M\tau}}^{\infty} w^{\frac{n}{2}-1} w^{-\frac{1}{2}} \exp(-w) dw dx.$$

The function $w^{-\frac{1}{2}}$ is decreasing. Replacing it by it's value at $\frac{2|x|^2}{M\tau}$ and using the inequality $w^{\frac{n}{2}-1} \exp(-w) \leq C \exp(\frac{-w}{2})$ we obtain

$$= \frac{C}{\tau^{\frac{n}{2}}} \int_{|x|<\epsilon} |x|^{-n+1} \int_{\frac{2|x|^2}{M\tau}}^{\infty} (\frac{2|x|^2}{M\tau})^{-\frac{1}{2}} \exp(-\frac{w}{2}) dw dx,$$

$$\leq \frac{C}{\tau^{\frac{n}{2}-\frac{1}{2}}} \int_{|x|<\epsilon} |x|^{-n} \exp(\frac{-|x|^2}{M\tau}).dx$$

Using polar coordinates, we obtain

$$\leq \frac{C}{\tau^{\frac{n}{2}-\frac{1}{2}}} \int_0^{\epsilon} \rho^{-n} \rho^{n-1} \exp(\frac{-\rho^2}{M\tau}) d\rho,$$

$$= \frac{C}{\tau^{\frac{n}{2}-\frac{1}{2}}} \frac{1}{\epsilon^2} \int_0^{\epsilon} \rho \exp(\frac{-\rho^2}{M\tau}),$$

$$\leq \frac{C}{\tau^{\frac{n}{2}-\frac{3}{2}}} \frac{1}{\epsilon^2} (1 - \exp(\frac{-\epsilon^2}{M\tau})).$$

The other terms can be bounded in a similar way.
The proof is complete. □

To complete the proof of theorem (4.1.2) we let

$$\epsilon^2 = E\tau, \tag{4.1.21}$$

where(the large constant) E will be chosen later. From lemma (4.1.3) we have

$$|I_1| \geq C^{-1}\tau^{-n}\int_0^{\epsilon}\rho^{n-1}\exp(\frac{-4\rho^2}{m\tau})\ d\rho.$$

Let $w = \frac{4\rho^2}{m\tau} \Rightarrow (\frac{m\tau w}{4})^{\frac{1}{2}} = \rho \Rightarrow dw = \frac{4}{m\tau}2\rho d\rho.$

By substituting in the above integral we get

$$|I_1| \geq C^{-1}\tau^{\frac{-n}{2}}\int_0^{\frac{4E}{m}} w^{\frac{n}{2}-1}\exp(-w)\ dw \geq C^{-1}\tau^{-\frac{n}{2}}. \tag{4.1.22}$$

From lemma (4.1.3) we have

$$|I_2| + |I_3| \leq C(\tau^{-\frac{n}{2}+1}\epsilon^{-2}\exp(\frac{-\epsilon^2}{M\tau}) + \epsilon\tau^{-\frac{n}{2}}),$$

$$= C(\tau^{-\frac{n}{2}+1}\epsilon^{-2}\exp(\frac{-E}{M}) + \epsilon\tau^{-\frac{n}{2}}).$$

But, $|I_1| \leq |I_2| + |I_3|$, then from (4.1.22) we have

$$C^{-1}\tau^{-\frac{n}{2}} \leq |I_1| \leq |I_2| + |I_3| \leq C(\tau^{-\frac{n}{2}+1}\epsilon^{-2}\exp(\frac{-E}{M}) + \epsilon\tau^{-\frac{n}{2}}).$$

So,

$$C^{-1}\tau^{-\frac{n}{2}} \leq C(\tau^{-\frac{n}{2}+1}\epsilon^{-2}\exp(\frac{-E}{M}) + \epsilon\tau^{-\frac{n}{2}}).$$

Using (4.1.21) again and multiplying both sides by $C\tau^{\frac{n}{2}}$, we obtain

$$1 \leq C^2(\tau\epsilon^{-2}\exp(\frac{-E}{M}) + \epsilon) \leq C^2(E^{-1} + \epsilon).$$

Now choose E so large that $E^{-1} < \frac{1}{4C^2}$ and $\epsilon < \frac{1}{4C^2}$ then

$$C^2(E^{-1} + \epsilon) < \frac{1}{2}.$$

Hence we obtain a contradiction $1 \leq \frac{1}{2}$.
The contradiction shows that $Q_1 \neq Q_2$ is wrong, so $Q_1 = Q_2$.

4.2 Regularization

In general, regularization is the approximation of an ill-posed problem by a family of neighbouring well-posed problems.

Also, regularization in mathematics refers to a process of introducing additional information in order to solve an ill-posed problem.

Many problems in the physical science can be expressed as mathematical models consisting of a linear system of equation such as

$$Ax = b \tag{4.2.1}$$

The inverse problem associated with this model is to compute the input x given some output b and the known model.

We want to approximate the best approximate-solution $x^{\dagger} = A^{\dagger}b$ (4.2.1) for a specific right-hand side b in the situation that the exact data b are not known, but that only an approximate b^{δ} with $\left|\left|b^{\delta} - b\right|\right|$ is available; we call b^{δ} the noisy data and δ the noise level.

We are looking for some approximate say x_{α}^{δ} of $x^{\dagger}$ which has two property: one it depends continuously on the (noisy) data b^{δ}, and the other one is that x_{α}^{δ} tends to $x^{\dagger}$ as the noise level δ decreases to zero and the regularization parameter α is chosen appropriately.

The choice rule of the regularization parameter α has to be linked with δ or b^{δ} and may be with other information about A or about.

Definition 4.2.1. [20] *Let $A : X \longrightarrow Y$ be abounded linear operator between the Hilbert spaces X and Y, $\alpha_0 \in (0, \infty]$. For every $\alpha_0 \in (0, \infty]$, let $R_{\alpha} : Y \longrightarrow X$ be a continuous (not necessarily linear) operator. The family R_{α} is called a regularization operator(for $A^{\dagger}$), if for all $y \in D(A^{\dagger})$, there exists a parameter choice rule $\alpha = \alpha(\delta, b^{\delta})$ such that*

$$\lim_{\delta \longrightarrow 0} \sup\{\left|\left|R_{\alpha(\delta,b^{\delta})} - b^{\delta}\right|\right| \backslash b^{\delta} \in Y, \left|\left|b^{\delta} - b\right|\right| \leq \delta\} = 0, \tag{4.2.2}$$

holds. Here $\alpha : R^+ \times Y \longrightarrow (0, \alpha_0)$ *is such that*

$$\lim_{\delta \longrightarrow 0} \sup\{\alpha(\delta, b^\delta) \backslash b^\delta \in Y, \left|\left|b^\delta - b\right|\right| \leq b^\delta\} = 0. \tag{4.2.3}$$

For a specific $y \in D(A^\dagger)$*, a pair* (R_α, α) *is called a (convergent) regularization method (for solving* $Ax = b$*) if (4.2.3)hold.*

One of the most important regularization types is the Tikhonov Regularization where in this method the minimization problem

$$\min_{x \in \mathbb{R}^n} \{||Ax - b||\},$$

is replaced by the least squares problem

$$\min_{x \in \mathbb{R}^n} \{||Ax - b||^2 + ||L_\alpha x||^2\},$$

where $L_\alpha \in \mathbb{R}^{k \times n}$, $k \leq n$, is called the regularization matrix. The simplest form of Tikhonov regularization takes $L_\alpha = \alpha I$ for some constant α. This choice of α gives us the minimization problem

$$\min_{x \in \mathbb{R}^n} \{||Ax - b||^2 + \alpha^2 \, ||x||^2\}.$$

4.3 Numerical solution for an inverse diffusion problem

In many applications, such as the heat conduction and hydrology, there is a need to recover the (possibly discontinuous) diffusion coefficient a from boundary measurements of solutions of a parabolic equation.
In this section a numerical solution for the nonlinear ill-posed diffusion problem is presented.

Consider the inverse problem of finding the pair (u, a) for the Cauchy problem

$$\frac{\partial u}{\partial t} - div(a \bigtriangledown u) = \delta(x - x^*)\delta(t) \quad on \quad \mathbb{R}^n \times (0, T), \tag{4.3.1}$$

$$u = 0 \qquad on \quad \mathbb{R}^n \times \{0\}, \tag{4.3.2}$$

where $a = a(x)$ is bounded and measurable, u is bounded and δ is the Dirac delta function.

Assume that $a = 1 + f$, where $f = 0$ outside a bounded region $\Omega \subset \mathbb{R}^n$ with piecewise C^2- smooth boundary $\partial\Omega$. As additional data the solution $u(x, t; x^*)$ given for $x,\ x^* \in \Omega^*,\ t \in (0, T)$ is considered. Here Ω^* is a bounded domain in $\mathbb{R}^n$ where $\bar{\Omega}^* \bigcap \bar{\Omega} = \phi$.

In solving (4.3.1)-(4.3.2) we have two difficulties to overcome:(1) nonlinearity which can be removed by replacing the original problem by it's linearization around constant coefficient. (2) ill-posedness: Due to illposedness the inverse problem (4.3.1)-(4.3.2) is not easy and it's numerical solution is possible mostly in one and two dimensional cases.

In this section C will denote (possibly) different constants depending only on n, Ω, and T. ν denotes the exterior unit normal to the boundary, $||\ ||_p\,(E)$the norm in $L_p(E)$ and $Q = \mathbb{R}^n \times (0, T)$.

Before we give a numerical solution for the problem (4.3.1)-(4.3.2), we start by linearizing it. Assume that $a = 1 + f,\ f = 0 \quad in \quad \mathbb{R}^n \backslash \Omega$, where $||f||_\infty\,(\Omega) \leq \epsilon$.

Let u_ϵ be the solution to (4.3.1), (4.3.2). Put $v_\epsilon = u_\epsilon - u_0$ and substitute for u_ϵ in (4.3.1), (4.3.2), we get

$$(u_{0t} + v_{\epsilon t}) - div((1 + f)\nabla(v_\epsilon + u_0)) = \delta(x - x^*)\delta(t) \quad on \quad Q,$$

$$v_\epsilon = 0 \quad on \quad \mathbb{R}^n \times \{0\}.$$

By simplifying the above equation we get:

$$u_{0t} + v_{\epsilon t} - \triangle v_\epsilon - \triangle u_0 - div(f\nabla v_\epsilon) - div(f\nabla u_0) = \delta(x - x^*)\delta(t) \quad on \quad Q, \quad (4.3.3)$$

$$v_\epsilon = 0 \quad on \quad \mathbb{R}^n \times \{0\}. \quad (4.3.4)$$

Let u_0 be the solution of the unperturbed problem:

$$u_{0t} - \triangle u_0 = \delta(x - x^*)\delta(t) \quad on \quad Q, \quad (4.3.5)$$

$$u_0 = 0 \quad on \quad \mathbb{R}^n \times \{0\}. \quad (4.3.6)$$

Then

$$v_{\epsilon t} - \triangle v_\epsilon = div(f\nabla v_\epsilon) + div(f\nabla u_0) \quad on \quad Q, \quad (4.3.7)$$

$$v_\epsilon = 0 \quad on \quad \mathbb{R}^n \times \{0\}. \tag{4.3.8}$$

We will show that for small ϵ the term $div(f\nabla v_\epsilon)$ is small relative to the term $div(f\nabla u_0)$. By dropping it we get the linearized equation

$$v_t - \triangle v = div(f\nabla u_0) \quad on \quad Q, \tag{4.3.9}$$

$$v = 0 \quad on \quad \mathbb{R}^n \times \{0\}. \tag{4.3.10}$$

The notation in [21] $|u| = ess\sup ||u(,t)||_2(\Omega) + ||\nabla_x u||_2(Q)$, where the sup is taken over $0 \le t \le T$ will be used.

lemma 4.3.1. [8] *Let* $||f||_\infty(\Omega) \le \epsilon$. *Then* $|v| \le C\epsilon$ *and* $||v - v_\epsilon|| \le C\epsilon^2$.

Proof. Subtracting (4.3.5), (4.3.6) from (4.3.3), (4.3.4), we get

$$v_{\epsilon t} - div(a\nabla v_\epsilon) = div(f\nabla u_0) \;\; on \;\; Q,$$

$$v_\epsilon = 0 \;\; on \;\; \mathbb{R}^n \times \{0\}.$$

It's well known that the solution u_0 to the problem (4.3.5), (4.3.6) is given by the formula

$$u_0(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} exp(-\frac{|x^* - x|^2}{4t}).$$

Since $x^* \in \Omega^*$, u_0 is smooth and continuous on $\bar{\Omega} \times [0,T]$, so

$$||f\nabla u_0||_2(Q) \le C\,||f||_\infty(\Omega) \le C\epsilon.$$

Using [21, $Theorem 2.1,\ p.143$](where $\psi_0 = 0,\ f = 0,\ f_j$ have to be replaced by $f\frac{\partial u_0}{\partial j}$), we get

$$||\,v_\epsilon\,|| \le C\,||f \bigtriangledown u_0||_2(Q) \le C\epsilon.$$

Now subtracting (4.3.9), (4.3.10) from (4.3.7), (4.3.8), we get

$$(v_\epsilon - v)_t - \nabla(v_\epsilon - v) = div(f\nabla v_\epsilon) \quad on \;\; \mathbb{R}^n \times (0,T),$$

$$v_\epsilon - v = 0 \quad on \;\; \mathbb{R}^n \times \{0\}.$$

Using the same theorem from [21] we obtain

$$||v_\epsilon - v|| \le C\,||f\nabla v_\epsilon||_2(Q) \le C\epsilon^2.$$

We know that

$$|v - v_\epsilon| \ge |(|v| - |v_\epsilon|)|\,,$$

then

$$|v| \le |v_\epsilon| + |v - v_\epsilon| \le C\epsilon.$$

The proof is complete. □

If f is a C^2-smooth compactly supported function, then an integral representation of a solution to the Cauchy problem (4.3.9), (4.3.10)is given by

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x,t;y,\tau) div(f \nabla u_0(y,\tau)) dy d\tau,$$

where

$$\Gamma(x,t;y,\tau) = \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} \exp\left(\frac{-|x-y|^2}{4(t-\tau)}\right), \quad x \in \mathbb{R}^n, \quad t > \tau. \tag{4.3.11}$$

is a fundamental solution of the heat equation.
Integrating by parts we obtain

$$v(x,t) = -\int_0^t \int_{\mathbb{R}^n} f(y,\tau) \nabla_y \Gamma(x,t;y,\tau). \nabla_y u_0(y,\tau) dy d\tau.$$

Now, using the formula (4.3.11) for Γ and the formula

$$u_0(y,\tau) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \exp\left(\frac{-|x^*-y|^2}{4\tau}\right).$$

We obtain the following integral representation

$$v(x,t;x^*) = \frac{-1}{4^{n+1}\pi^n} \int_0^t \int_\Omega f(y) \frac{(x-y).(x^*-y)}{(\tau(t-\tau))^{\frac{n}{2}+1}} exp\left(\frac{-|x-y|^2}{4(t-\tau)} - \frac{-|x^*-y|^2}{4\tau}\right) dy d\tau. \tag{4.3.12}$$

To reduce the overdeterminancy of the inverse problem, put $x = x^*$ and $t = T$. Hence, the linearized inverse problem is to find a function $f \in L_\infty(\Omega)$ given the function $F(x^*) = v(x^*, T; x^*)$, $x^* \in \Omega^*$.
From representation (4.3.12), this inverse problem is equivalent to the following integral equation

$$Af(x) = F(x), \quad x \in \Omega^*, \tag{4.3.13}$$

where $Af(x) = \int_\Omega k(x-y) f(y) dy$, and the kernel k is defined as

$$k(x) = \frac{-1}{4(4\pi)^n} \int_0^T \frac{|x|^2}{(\tau(T-\tau))^{\frac{n}{2+1}}} \exp\left(\frac{-|x|^2 T}{4\tau(T-\tau)}\right) d\tau. \tag{4.3.14}$$

A is considered as an operator from $L_2(\Omega)$ into $L_2(\Omega^*)$.

Equation (4.3.13) is a Fredholm integral equation of the first kind, which represents an (strongly) ill-posed problem because A maps any Sobolev space $H_k(\Omega)$ (with positive or negative k) into the space of functions analytic in a neighborhood of $\bar{\Omega}^*$.

To solve the ill-posed problem (4.3.13), the Tikhonov regularization will be used.
First, we need to replace the original equation (4.3.13) by the following one

$$(\alpha I + A^*A)f_\alpha = F^*, \tag{4.3.15}$$

where $F^* = A^*F$ and α is a regularization parameter.

Then, discretizing the regularized normal equation (4.3.15) and solving it numerically.
To find $A^* : L_2(\Omega^*) \longrightarrow L_2(\Omega)$, we have

$$(A\phi, \psi)_{L_2(\Omega^*)} = (\phi, A^*\psi)_{L_2(\Omega)}.$$

So

$$(A^*f^*)(y) = \frac{-1}{4(4\pi)^n}\int_0^T\int_{\Omega^*} f^*(x)\frac{|x-y|^2}{(\tau(T-\tau))^{\frac{3}{2}}}\exp(\frac{|x-y|^2\,T}{4\tau(T-\tau)})dxd\tau,\ \ y\in\Omega,$$

and we will have

$$(A^*Af)(x) = \frac{1}{16(4\pi)^{2n}}\int_{\Omega^*}\int_{\Omega} f(y)\,|y-z|^2\,|x-z|^2\,K(x-z,T)K(y-z,T)dydz,$$

where

$$K(x-z,T) = \int_0^T \frac{\exp(\frac{-|x-z|^2T}{4\tau(T-\tau)})}{(\tau(T-\tau))^{\frac{n+2}{2}}}d\tau.$$

We start with one-dimensional problem with $\Omega = [0,1]$, $\Omega^* = [2,3]$, $T = 1$.
So

$$(A^*Af)(x) = \frac{1}{(16\pi)^2}\int_2^3\int_0^1 f(y)\,|y-z|^2\,|x-z|^2\int_0^1\frac{\exp(\frac{|y-z|^2}{4(1-\tau)\tau})}{(\tau(1-\tau))^{\frac{3}{2}}}d\tau\int_0^1\frac{\exp(\frac{|x-z|^2}{4(1-\tau)\tau})}{(\tau(1-\tau))^{\frac{3}{2}}}d\tau dydz.$$

Discretizing by the trapezoid method we get

$$(A^{d*}A^df^d)(x_k) = \frac{1}{(16\pi)^2}\sum_{i=1}^n\sum_{j=1}^n f(y_j)\,|y_j-z_i|^2\,|x_k-z_i|^2\,(\sum_{m=1}^n\frac{\exp(\frac{|y_j-z_i|^2}{4(1-\tau_m)\tau_m})}{(\tau_m(1-\tau_m))^{\frac{3}{2}}}w(\tau_m))$$

$$(\sum_{m=1}^{n} \frac{\exp(\frac{|x_k - z_i|^2}{4(1-\tau_m)\tau_m})}{(\tau_m(1-\tau_m))^{\frac{3}{2}}} w(\tau_m))w(j)w(i).$$

Put

$$B(i,j,k) = \frac{1}{(16\pi)^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_j - z_i|^2 \, |x_k - z_i|^2 \, (\sum_{m=1}^{n} \frac{\exp(\frac{|y_j - z_i|^2}{4(1-\tau_m)\tau_m})}{(\tau_m(1-\tau_m))^{\frac{3}{2}}} w(\tau_m))$$

$$(\sum_{m=1}^{n} \frac{\exp(\frac{|x_k - z_i|^2}{4(1-\tau_m)\tau_m})}{(\tau_m(1-\tau_m))^{\frac{3}{2}}} w(\tau_m))w(j)w(i).$$

Hence, we have

$$\begin{aligned}
(A^{d*}A^d f^d)(x_1) = {} & B(1,1,1)f(y_1)+B(1,2,1)f(y_2)+B(1,3,1)f(y_3)+...+B(1,n,1)f(y_n) \\
& +B(2,1,1)f(y_1)+B(2,2,1)f(y_2)+B(2,3,1)f(y_3)+...+B(2,n,1)f(y_n) \\
& +B(n,1,1)f(y_1)+B(n,2,1)f(y_2)+B(n,3,1)f(y_3)+...+B(n,n,1)f(y_n). \\
(A^{d*}A^d f^d)(x_2) = {} & B(1,1,2)f(y_1)+B(1,2,2)f(y_2)+B(1,3,2)f(y_3)+...+B(1,n,2)f(y_n) \\
& +B(2,1,2)f(y_1)+B(2,2,2)f(y_2)+B(2,3,2)f(y_3)+...+B(2,n,2)f(y_n) \\
& +B(n,1,2)f(y_1)+B(n,2,2)f(y_2)+B(n,3,2)f(y_3)+...+B(n,n,2)f(y_n).
\end{aligned}$$

We continue until we get $k = n$, then we have

$$\begin{aligned}
(A^{d*}A^d f^d)(x_n) = {} & B(1,1,n)f(y_1)+B(1,2,n)f(y_2)+B(1,3,n)f(y_3)+...+B(1,n,n)f(y_n) \\
& +B(2,1,n)f(y_1)+B(2,2,n)f(y_2)+B(2,3,n)f(y_3)+...+B(2,n,n)f(y_n) \\
& +B(n,1,n)f(y_1)+B(n,2,n)f(y_2)+B(n,3,n)f(y_3)+...+B(n,n,n)f(y_n).
\end{aligned}$$

So,

$$\begin{bmatrix} (A^{d*}A^d f^d)(x_1) \\ (A^{d*}A^d f^d)(x_2) \\ . \\ . \\ . \\ (A^{d*}A^d f^d)(x_n) \end{bmatrix} =$$

$$\begin{bmatrix} B(1,1,1)+...+B(n,1,1) & B(1,2,1)+...+B(n,2,1) & ... & B(1,n,1)+...+B(n,n,1) \\ B(1,1,2)+...+B(n,1,2) & B(1,2,2)+...+B(n,2,2) & ... & B(1,n,2)+...+B(n,n,2) \\ . & & & \\ . & & & \\ . & & & \\ B(1,1,n)+...+B(n,1,n) & B(1,2,n)+...+B(n,2,n) & ... & B(1,n,n)+...+B(n,n,n) \end{bmatrix}$$
$$\begin{bmatrix} f(y_1) \\ f(y_2) \\ . \\ . \\ . \\ f(y_n) \end{bmatrix}.$$

We conclude that for $n-$dimensional problem, and after discretizing by the trapezoid method we get

$$A^{d*}A^d f^d = Bf^d,$$

where B is the matrix calculated from the discretization and f^d is a function of discrete argument defined on the rectangular uniform $N \times N$ grid.

Then the discretized equation (4.3.13)can be written as

$$(\alpha I + B)f^d = F^d.$$

An analytic calculation of the integral (4.3.12) is not realistic even for simplest f (say $f = 1$), therefore the data for the inverse problem is generated by numerical calculation using a similar discretization.
We will present a simple example to illustrate the algorithm described above. Assuming $n = 1$, $\Omega = [0, 2]$, $\Omega^* = [3, 5]$, $T = 4$, we recovered $f(x) = \sin(\frac{\pi x}{2})$ with $\alpha = 0.000001$.
Then, we obtain the following two figures using a matlab code.
Figure (4.1) shows the function $f(x)$.
Figure (4.2) illustrates recovery of $f(x)$ from exact data generated numerically.

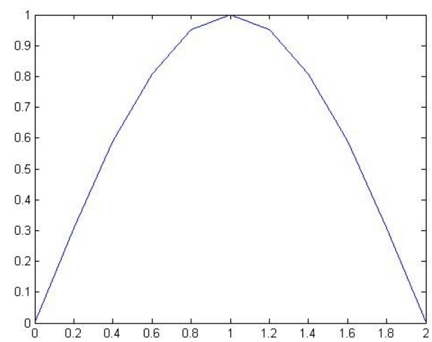

Figure 4.1: (a)

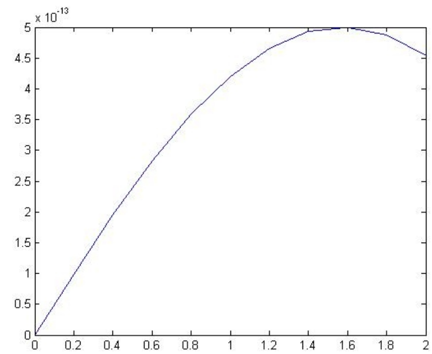

Figure 4.2: (b)

4.4 Regularization by the Laplacian operator

In the previous section an ill-posed, nonlinear inverse problem in heat conduction is studied. The problem is linearized to give a linear integral equation, and then it's solved by the Tikhonov method with the identity as the regularization operator.

It has been noted that the identity operator doesn't give good solution to the original equation in general. Hence, the Laplacian operator is used as a

regularization operator instead.
In the Laplacian approach for solving the illposed equation (4.3.13), the equation is replaced by the regularized equation

$$(\alpha\Delta + A^*A)f = A^*F,$$

where α is the regularization parameter and Δ is the Laplacian operator.

The regularized equations are solved by the conjugate gradient method with stopping criteria $\frac{||r_q||}{||r_0||} < 10^{-7}$, where r_q is the residual after the qth iteration. The initial guess is chosen to be the zero vector.

Numerical examples showed that the Laplacian operator is better than the identity operator in that the solutions obtained are more accurate.

However, the regularized equation becomes ill-conditioned when the first is used, and hence if the conjugate gradient method is used to solve the equation, the iteration number required for convergence will increase with the size of the discretization matrix. To speed up the convergence rate, the Laplacian itself is used as the preconditioner for the equation.

Chapter 5

Solving inverse parabolic problems using Adomian decomposition method

5.1 Adomian decomposition method

(ADM) is a well-known method for solving effectively, easily, and accurately a large class of linear and nonlinear, ordinary or partial differential equations.

The main advantage of the method is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or nonhomogeneous, with constant coefficients or with variable coefficients. Another important advantage is that the method is capable of greatly reducing the size of computation work and still has high accuracy.[16]

In this section we first introduce the Adomian Decomposition method for solving differential equations, then we implement a few applications of this method to both linear and nonlinear differential equations.

5.1.1 A general description of the ADM

We begin by considering the differential equation

$$Lu + Ru + Nu = g, \tag{5.1.1}$$

with prescribed conditions, where u is the unknown function, L is, mostly, the lower order derivative which is assumed to be invertible, R is other linear differential operator, Nu represent the nonlinear term and g is the source term.

Applying the inverse operator $L^{-1}$to both sides of equation (5.1.1), then we obtain

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu,$$

since (5.1.1) was taken to be a differential equation and L is linear, L^{-1} would represents an integration and with any given initial conditions, $L^{-1}Lu$ will give an equation for u using these conditions. This gives

$$u = f - L^{-1}[Ru + Nu], \tag{5.1.2}$$

where f represents the function generated by integrating g and using the initial conditions.
The nonlinear operator $[Nu]$ can be decomposed by an infinite series of polynomials given by

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n), \tag{5.1.3}$$

where $A_n(u_0, u_1, ..., u_n)$ is the appropriate Adomian's polynomials defined by

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}(N(\sum_{k=0}^{\infty} \lambda^k u_k))]_{\lambda=0}, \quad n > 0. \tag{5.1.4}$$

In Adomian decomposition method the solution of equation (5.1.1) is considered to be as the sum of an infinite series:

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t). \tag{5.1.5}$$

Substitution of (5.1.3) and (5.1.5) in (5.1.2) results in the following:

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}[R(\sum_{n=0}^{\infty} u_n) + \sum_{n=0}^{\infty} A_n].$$

Hence, the recursive relationship is found to be

$$u_0(x,y) = f(x,y),$$

$$u_{n+1}(x,y,t) = -L^{-1}[R(u_n) - A_n], \quad n \geq 0. \tag{5.1.6}$$

5.1.2 Applications

Problem 1:(Solving non homogeneous heat equation)[16] Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), \quad 0 < x < 1, \quad 0 < t < T,$$

subject to the initial condition

$$u(x,0) = f(x), \quad 0 \leq x \leq 1,$$

and the non-local boundary conditions

$$u(0,t) = \int_0^1 \varphi(x,t)u(x,t)dx + g_1(t), \quad 0 < t \leq T,$$

$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dx + g_2(t), \quad 0 < t \leq T,$$

where f, g_1, g_2, φ, ψ and q are known functions and are sufficiently smooth, T is given constant.

We begin by writing the problem in the standard form

$$L_t(u) = L_{xx}(u) + q(x,t), \tag{5.1.7}$$

where L_t and L_{xx} are given by

$$L_t(.) = \frac{\partial}{\partial t}(.) \quad \textit{and} \quad L_{xx}(.) = \frac{\partial^2}{\partial x^2}(.).$$

Assuming that the inverse operator L_t^{-1} exists and it's defined as

$$L_t^{-1} = \int_0^t (.)dt.$$

Applying the inverse operator on both sides of (5.1.7) and using the initial condition yields

$$u(x,t) = f(x) + L_t^{-1}(L_{xx}(u)) + L_t^{-1}(q(x,t)).$$

Now, we decompose the unknown function $u(x,t)$ by a sum of components defined by the series

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t),$$

where u_0 is defined as $u(x,0)$, the components $u_k(x,t)$ are obtained by the recursive formula

$$\sum_{k=0}^{\infty} u_k(x,t) = f(x) + L_t^{-1}\{L_{xx}(\sum_{k=0}^{\infty} u_k(x,t))\} + L_t^{-1}(q(x,t)).$$

Or

$$u_0(x,t) = f(x) + L_t^{-1}(q(x,t)),$$
$$u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u_k(x,t))), \quad k \geq 0.$$

Problem 2(Solving a nonlinear model)[17]
Consider the following hyperbolic nonlinear problem:

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1,$$

with the initial condition

$$u(x,0) = \frac{x}{10}, \quad 0 < x \leq 1.$$

Now, we use ADM to solve the problem. Then we have
$Nu = \psi(u) = u\frac{\partial u}{\partial x}$, $g(x,t) = 0$, $Ru = 0$, $L(u) = \frac{\partial u}{\partial t}$ and $f = u(x,0) = \frac{x}{10}$.

The Adomian's polynomials can be derived using (5.1.4) as follows:

$$A_0 = u_0\frac{\partial u_0}{\partial x},$$

$$A_1 = u_1\frac{\partial u_0}{\partial x} + u_0\frac{\partial u_1}{\partial x},$$

$$A_2 = u_2\frac{\partial u_0}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_0\frac{\partial u_2}{\partial x},$$
$$A_3 = u_3\frac{\partial u_0}{\partial x} + u_2\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_2}{\partial x} + u_0\frac{\partial u_3}{\partial x},$$

and so on.

By using the recursive formulas (5.1.6), we have

$$u_0 = \frac{x}{10},$$

$$u_1 = \frac{x}{10}(\frac{t}{10}),$$
$$u_n = \frac{x}{10}(\frac{t}{10})^n.$$

Substituting these individual terms in the infinite series we obtain

$$u(x,t) = \frac{x}{10}[1 + \frac{t}{10} + (\frac{t}{10})^2 + ... + (\frac{t}{10})^n].$$

5.2 Solution of some parabolic inverse problems by ADM

In this section two types of parabolic inverse problems are studied by Adomian decomposition method.

This method eliminates the need to solve any linear or nonlinear system of algebraic equations, also numerical results obtained from this method indicate high accuracy and speed of the method.

5.2.1 Parabolic inverse problems with unknown boundary conditions

Consider the following inverse parabolic problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad 0 < t < T_0, \tag{5.2.1}$$

subject to the following initial and boundary conditions:

$$u(x,0) = f(x), \quad 0 \le x \le 1, \quad 0 \le t < T_0,$$

$$u(0,t) = g(t), \quad 0 \le x \le 1, \quad 0 \le t < T_0,$$

$$\frac{\partial u}{\partial x}(1,t) = h(t), \quad 0 \le x \le 1, \quad 0 \le t < T_0, \tag{5.2.2}$$

where T_0 is constant, f is continuous known function, g is infinitely differentiable function, the temperature $u(x,t)$ is unknown, the heat flux $\frac{\partial u}{\partial x}(0,t)$ and $g(t)$ are remained to be determined.

To determine the a mount of flux, an extra condition

$$u(x_1,t) = K_1(t), \tag{5.2.3}$$

is used where $K_1(t)$ is a known continuous function and x_1 is an internal point where the sensors are located.

We start by dividing problem (5.2.1)-(5.2.3) into two separated problems: The direct problem, which is defined in $\{(x,t)\backslash x_1 < x < 1,\ 0 < t < T_0\}$ is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x_1 < x < 1, \quad 0 < t < T_0, \tag{5.2.4}$$

$$u(x,0) = f(x), \quad 0 < x_1 < x < 1, \quad 0 < t < T_0,$$

$$u(x_1,t) = K_1(t), \quad 0 < x_1 < x < 1, \quad 0 < t < T_0, \tag{5.2.5}$$

and the inverse problem, which is located in $\{(x,t) : 0 < x < x_1,\ 0 < t < T_0\}$ is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < x_1, \quad 0 < t < T_0, \tag{5.2.6}$$

$$u(x,0) = f(x), \quad 0 < x < x_1, \quad 0 < t < T_0,$$

$$u(x_1,t) = K_1(t), \quad 0 < x < x_1, \quad 0 < t < T_0. \tag{5.2.7}$$

Now, integrating both sides of equation (5.2.6) w.r.t x, so that an approximate value of $u_x(0,t)$ is obtained. Therefore, we have

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(x_1,t) - \int_0^{x_1} \frac{\partial u}{\partial t} dx. \tag{5.2.8}$$

To find $u(0,t)$, we can integrate both sides of equation (5.2.6) twice w.r.t x. Thus, we have

$$u(0,t) = u(x_1,t) - \int_0^{x_1}\int_0^{x} \frac{\partial u}{\partial t} dx dx. \tag{5.2.9}$$

In both relations (5.2.8), (5.2.9) the a mount $u(x,t)$ is unknown.

Using the direct problem(5.2.4)-(5.2.5), we can find the values of $u(x,t)$. Therefore, integrating both sides of equation (5.2.4)w.r.t t we get:

$$u(x,t) = u(x,0) + \int_0^t \frac{\partial^2 u}{\partial x^2} dt = f(x) + \int_0^t \frac{\partial^2 u}{\partial x^2} dt. \tag{5.2.10}$$

Put $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ in (5.2.10), we will have:

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + \int_0^t \frac{\partial^2(\sum_{n=0}^{\infty} u_n(x,t))}{\partial x^2} dt. \tag{5.2.11}$$

Hence we find:

$$u_0(x,t) = f(x),\ \ u_1(x,t) = \int_0^t \frac{\partial^2(u_0(x,t))}{\partial x^2} dt = tf''(x),\ \ u_2(x,t) = \frac{t^2}{2!} f^{(4)}(x), ... \tag{5.2.12}$$

and so on the approximate values of $u(x,t)$ are obtained.

Finally, substituting the approximate values of $u(x,t)$ in formula (5.2.8) and (5.2.9), to obtain the approximate value of $\frac{\partial u(0,t)}{\partial x}$ and $u(0,t)$.

Theorem 5.2.1. [9]*(Existence and uniqueness) Let f and h be two continuous functions in their range, then the function:*

$$u(x,t) = w(x,t) - 2\int_0^t \theta(x,t-\tau)g(\tau)d\tau + 2\int_0^t \theta(x-1,t-\tau)h(\tau)d\tau,$$

where

$$w(x,t) = \int_0^1 (\theta(x-\zeta,t) + \theta(x+\zeta,t))f(\zeta)d\zeta,$$

and

$$\theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m,t),$$

and

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(\frac{-x^2}{4t}),$$

is a unique solution of problem (5.2.1)-(5.2.3).

5.2.2 Inverse parabolic problem with unknown control function

Consider the following inverse heat radiation problem :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a(t)u, \quad (x,t) \in D, \tag{5.2.13}$$

subject to the initial and boundary conditions

$$u(x,0) = f(x), \quad x \geq 0, \quad 0 \leq t \leq T,$$

$$u(0,t) = h(t);\ h(t) \neq 0, \quad x \geq 0, \quad 0 \leq t \leq T, \tag{5.2.14}$$

where $D = \{(x,t) : x > 0,\ 0 < t < T\}$, $T > 0$ is a constant and $f,\ h$ are smooth functions on their range.

The functions $a(t)$ and $u(x,t)$ are unknown. Thus we need an extra additional condition:

$$-\frac{\partial u}{\partial x}(0,t) = g(t), \quad 0 \leq t \leq T. \tag{5.2.15}$$

We start by transform problem (5.2.13)-(5.2.15) using the conversion:

$$v(x,t) = r(t)u(x,t);\ r(t) = \exp(-\int_0^t a(s)ds). \tag{5.2.16}$$

So, equations (5.2.11)-(5.2.13) becomes:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \tag{5.2.17}$$

$$v(x,0) = f(x), \quad x \geq 0, \quad 0 \leq t \leq T,$$

$$v(0,t) = p(t), \quad x \geq 0, \quad 0 \leq t \leq T, \tag{5.2.18}$$

subject to the extra additional condition:

$$-\frac{\partial v}{\partial x}(0,t) = q(t), \quad 0 \leq t \leq T, \tag{5.2.19}$$

where $q(t) = r(t)g(t)$ and $p(t) = r(t)h(t)$.

To, solve problem (5.2.13)-(5.2.14) by Adomian Decomposition method, we first solve problem (5.2.17)-(5.2.18)by the mentioned method.

Therefore, we integrate both sides of (5.2.17)-(5.2.18) w.r.t t then we have

$$v(x,t) = v(x,0) + \int_0^t \frac{\partial^2 v}{\partial x^2}(x,t)dt = f(x) + \int_0^t \frac{\partial^2 v}{\partial x^2}(x,t)dt.$$

By fixing $v = \sum_{n=0}^{\infty} v_n$ in the above relation, we obtain:

$$\sum_{n=0}^{\infty} v_n(x,t) = f(x) + \int_0^t \frac{\partial^2 (\sum_{n=0}^{\infty} v_n(x,t))}{\partial x^2} dt.$$

Hence, we have

$$v_0(x,t) = f(x),\ v_1(x,t) = \int_0^t \frac{\partial^2 v_0}{\partial x^2} dt = t f^{''}(x),\ v_2(x,t) = \frac{t^2}{2!} f^{(4)}(x), ...$$

Consequently, the approximate value of $v(x,t)$ are obtained.

Then, by using the over extra additional condition (5.2.19), we calculate $r(t)$, and we calculate $a(t)$ by using:

$$a(t) = -\frac{r^{'}(t)}{r(t)}.$$

Also, by replacing the approximate value of $v(x,t)$ and the value of $r(t)$ in (5.2.16), we obtain $u(x,t)$.

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